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THE DIRICHLET SERIES FOR POWERS OF MAPS ON NATURAL NUMBERS

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Abstract: In this paper we study the Dirichlet series of natural powers of maps with $O_n(T) = n^a$, $n \in \mathbb{N}$ and a is a nonnegative integer and we also find the abscissa of convergence of Dirichlet series.

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Key Words: orbit Dirichlet series, Riemann zeta function, abscissa of convergence

1. Introduction

A *Dirichlet series* is any series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where s and a_n are complex numbers, $n = 1, 2, 3, \cdots$. If $a_n = 1$ for all n then the Dirichlet series is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is the Riemann zeta function, see details in [1].

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In 2009, Pakapongpun, A. and Ward, T. [3] proposed the orbit Dirichlet series

$$d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s}$$

if T is a continuous mapping $T : X \to X$, when X is a compact metric space and $O_n(T) = n^a$ for all $n \geq 1$ and for some integer $a \geq 0$.

In 2011, Pakapongpun, A. studied orbit Dirichlet series of a prime power of maps $d_{T^p}(s)$ where $O_n(T) = n^a$ and p is a prime number, see all detail in [2].

In 2012,Rakporn Dokchan and Apisit Pakapongpun studied the special cases of the number theoretic of Dirichlet series, the detail is in [4].

In this paper has been improved to the general form of Dirichlet series for powers of maps $d_{T^m}(T)$ where m is a positive integer.

2. Preliminary Notes

Definition 2.1. Let T be a map. A *closed orbit* τ of length $|\tau|$ is a set of the form

$$\{x, Tx, T^2x, \cdots, T^{|\tau|}x = x\}$$

with cardinality $|\tau|$. The number of points of period n is

$$F_n(T) = \sum_{d|n} dO_d(T),$$

where $O_n(T)$ is the number of closed orbits of length n under T. and assume $O_n(T) < \infty$ for all $n \geq 1$. From Möbius inversion formula

$$O_n(T) = \frac{1}{n} \sum_{d|n} \mu(\frac{n}{d}) F_d(T)$$

which is $\mu : \mathbb{N} \to \mathbb{R}$ is the Möbius function.

The orbits of length n under the iterate T^p for some prime p and T^m for integer m are shown in the next two theorem, see more detail in [1].

The number of orbits of length n under an iterate T^p for some prime p will be shown in the next theorem, see the proof theorem 3.1 in [5].

Theorem 2.2. *Let p be a prime. Then*

$$O_n(T^p) = \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T)) & \text{if } p \mid n. \end{cases}$$

Theorem 2.3. *Let $m = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ and $S = \{p_1, p_2, \cdots, p_t\}$, let D_n be the set of p such that $p \in S$ and $p_i \mid n$ for each n. Define r_p is the maximum power k such that $p^k \mid m$. If $O_n(T) = n^k, p_1 p_2 \cdots p_j \nmid n$ and $p_{j+1} p_{j+2} \cdots p_t \nmid n$ then*

$$O_n(T^{p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}}) = p_{j+1}^{r_{j+1}} \dots p_t^{r_t} \sum_{\substack{0 \le i_1 \le r_1 \\ \vdots \\ 0 \le i_j \le r_j}} p_1^{i_1} \dots p_j^{i_j} O_{np_1^{i_1} \dots p_j^{i_j} p_{j+1}^{r_{j+1}} \dots p_t^{r_t}}(T)$$

see detail the proof theorem 3.2 in [5].

3. Main Results

Theorem 3.1. *Let $m = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ and $S = \{p_1, p_2, \cdots, p_t\}$. For each n let D_n be the set of p such that $p \in S$ and $p_i \mid n$. Define r_p is the maximum power k such that $p^k \mid m$. If $O_n(T) = n^k$ then*

$$d_{T^m}(s) = \zeta(s-k) \prod_{p \in S} \left((1 + p^{2(k+1)} + \cdots + p^{(k+1)(r_p-1)}) \left(1 - \frac{1}{p^{s-k}}\right) + p^{(k+1)r_p} \right)$$

and the abscissa of convergence d_{T^m} is $k+1$.

Proof. From theorem 2.3, we have

$$\sum_{n=1}^{\infty} \frac{O_n(T^m)}{n^s} =$$

$$\sum_{n=1}^{\infty} \frac{\left(\prod_{p \in D_n} p^{r_p}\right) \sum_{\substack{p \in S - D_p \\ 0 \le i_p \le r_p}} \left(\prod_{p \in S - D_n} p^{i_p}\right) O_{n(\prod_{p \in S - D_n} p^{i_p})(\prod_{p \in D_n} p^{r_p})}(T)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{\left(\prod_{p \in D_n} p^{r_p}\right) \sum_{\substack{p \in S - D_p \\ 0 \le i_p \le r_p}} \left(\prod_{p \in S - D_n} p^{i_p}\right) \prod_{p \in S - D_n} p^{k i_p} \prod_{p \in D_n} p^{k r_p}}{n^{s-k}}$$

$$= \sum_{n=1}^{\infty} \frac{\sum_{\substack{p \in S - D_n \\ 0 \le i_p \le r_p}} \prod_{p \in S - D_n} p^{(k+1) i_p} \prod_{p \in D_n} p^{(k+1) r_p}}{n^{s-k}}$$

$$= \sum_{n=1}^{\infty} \frac{\prod_{p \in S - D_n} (1 + p^{2(k+1)} + \cdots + p^{(k+1) r_p}) \prod_{p \in D_n} p^{(k+1) r_p}}{n^{s-k}}$$

$$= \prod_{p \notin S} \left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)$$
$$\prod_{p \in S} \left((1 + p^{2(k+1)} + \dots + p^{(k+1)r_p}) + \frac{p^{(k+1)r_p}}{p^{s-k}} + \frac{p^{(k+1)r_p}}{p^{2(s-k)}} + \dots\right)$$
$$= \prod_{p \notin S} \left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)$$
$$\prod_{p \in S} \left((1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)}) + p^{(k+1)r_p}\left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)\right)$$

Now consider the product above. Let $T \subset S$. The expression above is equal to

$$\sum_{T \subset S} \prod_{p \in T} (1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)})$$
$$\prod_{p \in S-T} p^{(k+1)r_p} \left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)$$
$$\prod_{p \notin S} \left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)$$
$$= \sum_{T \subset S} \prod_{p \in T} (1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)}) \left(1 - \frac{1}{p^{s-k}}\right)$$
$$\prod_{p \in S-T} p^{(k+1)r_p} \prod_{p} \left(1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \dots\right)$$
$$= \zeta(s-k) \sum_{T \subset S} \prod_{p \in T} (1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)}) \left(1 - \frac{1}{p^{s-k}}\right) \prod_{p \in S-T} p^{(k+1)r_p}$$
$$= \zeta(s-k) \prod_{p \in S} \left((1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)}) \left(1 - \frac{1}{p^{s-k}}\right) + p^{(k+1)r_p}\right).$$

Here $\prod_{p \in S} \left((1 + p^{2(k+1)} + \dots + p^{(k+1)(r_p-1)}) \left(1 - \frac{1}{p^{s-k}}\right) + p^{(k+1)r_p}\right)$ is absolutely convergent for any value s. So, the abscissa of convergence d_{T^m} is $s = k+1$ □

Example 3.2. Let $d_T(s) = \zeta(s-2) = \sum_{n=1}^{\infty} \frac{1}{n^{s-2}}$. Then

$$d_{T^{10}}(s) = \zeta(s-2)(9 - \frac{1}{2^{s-2}})(123 - \frac{1}{5^{s-2}}),$$

it is obvious, the abscissa of convergence $d_{T^{10}}$ is 3

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