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A PRESENTATION OF $M_{n,m}$

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Abstract: Let $M_{n,m}$ be a free metabelian nilpotent Lie algebra of rank n and nilpotency class $m-1$. We show that $M_{n,m}$ admits a minimal presentation whose set of defining relators is the set of all basic commutators of length m.

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1. Introduction

Presentations and minimal presentations are special classes in the theory of groups and algebras. The definition of presentation of a group was introduced by D. L. Johnson in [4] and [5]. If generators and defining relations of a presentation of a group are finite then it is called a finitely presented group. Such a presentation is called minimal if the set of defining relations is the smallest subset with this property. Wamsley and Searby emphasized minimal presentations of groups in [3].Then, Wamsley obtained some results about minimal presentations for finite groups in [6]. In [8], Moghaddam considered these results for free groups. Gerdt and Kornyak examinated this presentations of groups in Lie algebras. The aim of this work is to find a presentation for a free metabelian nilpotent Lie algebra of rank $n \geq 2$. We give a presentation of a free metabelian nilpotent Lie algebra of rank n and nilpotency class $m-1$ (for more information about this topic can be reached in [1], [2] and [9]).

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2. Preliminaries

Let F be a free Lie Algebra generated by a set $X = \{x_1, x_2, ..., x_n\}$ over a field K of characteristic 0 and $M_{n,m}$ be a free metabelian Lie algebra of nilpotency class $m-1$ and rank n. Denote by F' and $\gamma_m(F)$ the derived subalgebra and the $m-th\ term$ of the lower central series of F respectively. We identify $M_{n,m}$ with $F/\gamma_m(F) + F''$ in the usual way, where F'' is the derived subalgebra of F'. Consider the Lie algebra defined by the presentation $M_{n,m} = (x, y \mid \gamma_m(F) + F'')$. Then $M_{n,m} \cong F/\gamma_m(F) + F''$.

We write (ab) for the Lie product of two elements a and b. A word of length n is an ordered $n - tuples$ of the elements of X. We write $\ell(u)$ for the length of the word $u \in F$.

Definition 1. A Hall Basis H of F is defined as follows:

i) $X \subset H$ and X is given an arbitrary total ordering. Let $H_1 = X$.

ii) H_2 consists of element (xy) such that $x, y \in X$ and $x > y$,

$$H_2 = \{(xy) \mid x, y \in X;\ x > y\}.$$

iii) $Hn = \{h = ((uv)\,w) \mid u, v, w \in \cup_{i=1}^{n-1} H_i; u > v \leq w, \ell(h) = n\}$.

Now, we put $H = \cup_{n=1}^{\infty} H_n$. Then, H is the Hall basis of F constructed on X. We fix the following ordering of the set H :

1. X is ordered arbitrarily. (usually consistent with the indices used to denote its letters)

2. Let $u = u_1u_2$, $v = v_1v_2$ be elements of H. If $length(u) > length(v)$, put $u > v$. If u and v have the same length, then put $u > v$ if and only if either $u_1 > v_1$ or $u_1 = v_1$ and $u_2 > v_2$. Free generating sets for the lower central terms of a free Lie algebra are given by A. L. Šmel'kin in [10].

Theorem 2. (see [10]) *The set C_m defined as*

$$Cm = \{x = (a_1a_2) \mid x, a_1, a_2 \in H;\ \ell(x) \geq m; \ell(a_2) < m\}$$

is a set of free generators for $\gamma_m(F)$.

We construct a Hall basis H^{C_m} on C_m for $\gamma_m(F)$, by forming products of elements of C_m. If h is a product of elements from C_m, we will refer $C_m - \ell$ and $X - \ell$ of h meaning the number of letters used from C_m or X respectively. C_m

is a subset of H, so it can be given an order which coincides with the order in H. Let

$$\begin{aligned} H_1^{C_m} &= C_m. \\ H_2^{C_m} &= \{x = (a_1 a_2) : a_1, a_2 \in C_m; C_m - \ell(x) = 2\}. \end{aligned}$$

We now give $H_2^{C_m}$ an order as follows: Let $h, g \in H_2^{C_m}$, where $h = h_1 h_2$ and $g = g_1 g_2$, $h_1, h_2, g_1, g_2 \in C_m$. If $X - \ell(h)$ is less than $X - \ell(g)$, put $h < g$. Suppose h and g have the same $X - \ell(h)$. Then put $h < g$ if and only if either $h_1 < g_1$ or $h_1 = g_1$ and $h_2 < g_2$. Suppose $H_1^{C_m}, ..., H_{n-1}^{C_m}$ are defined and ordered.We put,

$$\begin{aligned} H_n^{C_m} &= \{x = ((a_1 a_2) a_3) \mid C_m - \ell(x) = n; \\ & \quad a_1, a_2, a_3, (a_1 a_2) \in \bigcup_{i=1}^{n-1} H_i^{C_m}; a_1 > a_2, a_2 \leq a_3\}. \end{aligned}$$

where the inequality signs refer to the ordering in $\bigcup_{i=1}^{n-1} H_i^{C_m}$.

We defined a free generating set C_m for F_m. We then constructed a Hall basis H^{C_m} for F_m. Consider F_{m_1,m_2} as a free subalgebra of F_m. Proceeding in the same manner we define a free generating set C_{m_1,m_2} as follows:

$$\begin{aligned} C_{m_1,m_2} &= \{x = (a_1 a_2) \mid x \in H^{C_{m_1}}; C_{m_1} - \ell(x) \geq m_2; \\ & \quad C_{m_1} - \ell(a_2) < m_2\}. \end{aligned}$$

We define C_{m_1,m_2} as follows:

Let $g, h \in C_{m_1,m_2}$. We set $g < h$, if $C_{m_1} - \ell(g) < C_{m_1} - \ell(h)$.

If $C_{m_1} - \ell(g) = C_{m_1} - \ell(h)$ and $X - \ell(g) < X - \ell(h)$, then again we set $g < h$.

Suppose both $C_{m_1} - \ell(g) = C_{m_1} - \ell(h)$ and $X - \ell(g) = X - \ell(h)$. Then we set $g < h$ if either $g_1 < h_1$ or $g_1 = h_1$ and $g_2 < h_2$ where $g = g_1 g_2$ and $h = h_1 h_2$.

3. A Presentation of $M_{n,m}$

Let F be a free Lie algebra generated by the set $X = \{x_1, x_2, ..., x_n\}$ over a field K of characteristic zero. We define the folowing subset of F :

$$\begin{aligned} C_{2,2,c} &= \{x = ((a_1 a_2) a_3) \mid C_2 - \ell(a_3) = 1; \ell(x) = c \\ & \quad a_1, a_2, a_3 \in H^{C_2}; a_1 > a_2, a_2 \leq a_3\}. \end{aligned}$$

$$S_{n,m} = C_m \cup (\bigcup_{c=4}^{m-1} C_{2,2,c})$$

In this section , we shall give a presentation of $M_{n,m}$. We have that $M_{n,m} = (X \mid \gamma_m(F) + F'')$. We define an order on X by $x_1 > x_2 > ... > x_n$. In the rest of the paper we will use this ordering. For any subset B of F by $\langle B \rangle$ we denote the ideal F of generated by B.

Theorem 3. *For $n \geq 2$ and $m \geq 5$, the free metabelian nilpotent Lie algebra $M_{n,m}$ admits the following presentation*

$$M_{n,m} = (X \mid S_{n,m})$$

Proof. We are going to obtain the proof in three steps:

i) $\gamma_m(F) \subseteq \langle C_m \rangle$

ii)$\langle\, C_{2,2} \setminus \bigcup_{c=4}^{m-1} C_{2,2,c} \rangle \subseteq \langle C_m \rangle$

iii) $\langle C_{n,m} \rangle + \langle C_{2,2,c} \rangle = \langle S_{n,m} \rangle$

i) Let $\langle C_m \rangle = I$ be the ideal of F which is generated by C_m. Firstly, we are going to prove that every element in $\gamma_m(F)$ are belong to the ideal of generated by C_m.

Let $u \in \gamma_m(F)$. Then, the form of u is $\sum_i \alpha_i \omega_i$ where $\omega_i \in C_m$, $\alpha_i \in F$. Thus, the form of ω_i is $(((c_1 c_2) \ldots) c_k)$ such that $c_j \in C_m$. Since, C_m is free generators set, we get $\omega_i \in I$. Thus, $u \in I$. This is show that, $\gamma_m(F) \subset \langle C_m \rangle$. Hence,

$$M_{n,m} = \left(X \mid \gamma_m(F) + F''\right) = \left(X \mid \langle C_m \rangle + F''\right).$$

ii) Let $u \in \left\langle C_{2,2} \setminus \bigcup_{c=4}^{m-1} C_{2,2,c} \right\rangle$. Then, for $c \geq m$, $u \in C_{2,2,c}$.Thus, the form of u is $\omega_1 \alpha_1 + ... + \omega_k \alpha_k$, where $\omega_i \in \langle C_{2,2,c} \rangle$, $\alpha_i \in F$. We are going to show that $\omega_i \alpha_i \in \langle C_m \rangle$. That is, we are going to show that $w\alpha \in \langle C_m \rangle$.

Let $w = (ab)$ such that $a > b$ and $C_2 - \ell(b) = 1$. Let $b = (((x_i x_{i_2}) \ldots) x_{i_k})$ where $\ell(b) = k$. Considering that, since $\ell(w) \geq m$ and $\ell(w) = \ell(a) + \ell(b)$ then, $\ell(a) + k \geq m$. Hence , $\ell(a) \geq m - k$. Thus, we consider the following two cases:

Case 1) If $1 < k \leq m - 1$ then since $\ell(b) < m$, we get $w = (ab) \in C_m$.

Case 2) If $\ell(b) \geq m$ then, we consider the following as:

Let $u = (((x_{i1}x_{i2})\ldots)x_{ik-1})$.Then, $w = (a(ux_{ik}))$. Hence, w can be written as (using the Jacobi identity)

$$w = (-u(x_{ik}a)) - (x_{ik}(au)) = (-(ax_{ik})u) + ((au)x_{ik}).$$

Since, $\ell(a) \geq \ell(b)$ then $\ell(a) \geq k \geq m$, the first term in the above sum. Hence, $-((ax_{ik})u) \in \langle C_m \rangle$.

In the second term in the above sum, since $\ell(b) > \ell(\alpha)$, we get $\ell(a) > \ell(\alpha)$. Applying the Jacobi identity of consecutive, we obtain $((au)x_{ik}) \in \langle C_m \rangle$. Thus, $w \in \langle C_m \rangle$. Hence, $u \in \langle C_m \rangle$. This is show that, $\left\langle C_{2,2} \setminus \bigcup_{c=4}^{m-1} C_{2,2,c} \right\rangle \subseteq \langle C_m \rangle$. Hence,

$$M_{n,m} = \left(x \mid \langle C_m \rangle + \langle \bigcup_{c=4}^{m-1} C_{2,2,c} \rangle \right).$$

iii) We are going to show that the equality $\langle C_m \rangle + \langle \bigcup_{c=4}^{m-1} C_{2,2,c} \rangle = \langle S_m \rangle$. Let $\langle C_m \rangle = I_1$, $\langle \bigcup_{c=4}^{m-1} C_{2,2,c} \rangle = I_2$, $\langle S_{n,m} \rangle = I$. Then, I can be written as $I = I_1 + I_2$. Let $u_1 + u_2 \in I_1 + I_2$ such that $u_1 \in I_1$ and $u_2 \in I_2$. Then, the form of u_1 is $\sum_i g_{i1}v_{i1}$ where $g_{i1} \in C_m$, $v_{i1} \in F$ and the form of u_2 is $\sum_i g_{i2}v_{i2}$ where $g_{i2} \in \bigcup_{c=4}^{m-1} C_{2,2,c}$, $v_{i2} \in F$. Hence, we get $u_1 + u_2 = \sum_i (g_{i1}v_{i1} + g_{i2}v_{i2})$. Since, $g_{i1}, g_{i2} \in C_m \cup \left(\bigcup_{c=4}^{m-1} C_{2,2,c} \right)$ then $g_{i1}, g_{i2} \in S_{n,m}$. Thus, we obtain $u_1 + u_2 \in I$. Hence, $I_1 + I_2 \subseteq I$.

Now, we are going to show that $I \subseteq I_1 + I_2$. Let $u \in I$ such that $u = \sum w_i\alpha_i$, $w_i \in S_{n,m}$. Then, we can be said a part of ω_i is in the C_m and other part of ω_i is in the $\bigcup_{c=4}^{m-1} C_{2,2,c}$ in the above sum. Suppose that $\omega_i \in C_m$ for $1 \leq i \leq r$ and $\omega_i \in \bigcup_{c=4}^{m-1} C_{2,2,c}$ for $i > r$. Then, u can be written as $u = \sum_{i=1}^{r} w_i\alpha_i + \sum_{i=r} w_i\alpha_i$. Hence, we get $u \in I_1 + I_2$. Thus, $I \subseteq I_1 + I_2$. This is show that $I = I_1 + I_2$. That is;

$$\langle C_m \rangle + \langle \bigcup_{c=4}^{m-1} C_{2,2,c} \rangle = \langle S_m \rangle.$$

Therefore, $M_{n,m}$ admits the following presentation

$$M_{n,m} = (X \mid \langle S_m \rangle) = (X \mid S_{n,m}).$$

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