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ON THE SLOPE OF THE SCHUR FUNCTOR OF A VECTOR BUNDLE

Elena Rubei
Dipartimento di Matematica e Informatica "U. Dini"
Viale Morgagni 67/A, 50134, Firenze, ITALY

Abstract: We prove that, for any complex vector bundle E of rank e on a compact Kähler manifold X, we have that $\mu(S^\lambda E) = |\lambda|\ \mu(E)$ for any $\lambda = (\lambda_1, ..., \lambda_{e-1})$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq ... \geq \lambda_{e-1}$, where $|\lambda| = \lambda_1 + ... + \lambda_{e-1}$, the symbol S^λ denotes the Schur functor and μ is the slope. This result has already been stated, without proof, by Ottaviani in 1995.

AMS Subject Classification: 19L10, 55R10
Key Words: slope, Schur functors

1. Introduction

In this short note we prove that for any complex vector bundle E of rank e on a compact Kähler manifold X, we have that

$$\mu(S^\lambda E) = |\lambda|\ \mu(E)$$

for any $\lambda = (\lambda_1, ..., \lambda_{e-1})$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq ... \geq \lambda_{e-1}$, where $|\lambda| = \lambda_1 + ... + \lambda_{e-1}$, the symbol S^λ denotes the Schur functor and μ is the slope. This result has already been stated, without proof, in [3] (and used in the recent paper [4]) In [1] it is stated and proved only in the case of the exterior powers.

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Since the lack of a proof of the general case in the literature, we expose here a complete proof.

Among the literature on the subject, we quote also [5]: in it, the author calculated the Chern characters of the symmetric powers and of the exterior powers of a vector bundle. It seems difficult to generalize his results to any Schur functors of a vector bundle.

2. Notation and Recalls

Let $\lambda = (\lambda_1, ..., \lambda_{e-1})$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq ... \geq \lambda_{e-1}$. For any V complex vector space of dimension n, the symbol $S^\lambda V$ will denote the Schur representation ($SL(V)$-representation) associated to λ (see Lecture 6 in [2]).

The $S^\lambda V$ are irreducible $SL(V)$-representations and it is well-known that all the irreducible $SL(V)$-representations are of this form.

We recall that Pieri's formula says that, if $\nu = (\nu_1, \nu_2, ...)$ is a partition of a natural number d with $\nu_1 \geq \nu_2 \geq ..$ and t is a natural number, then

$$S^\nu V \otimes S^t V = \oplus_{\gamma \in \Gamma} S^\gamma V$$

as $SL(V)$-representation, where Γ is the set of all the partitions $\gamma = (\gamma_1, \gamma_2, ...)$ with $\gamma_1 \geq \gamma_2 \geq ...$ of $d + t$ whose Young diagrams are obtained from the Young diagram of ν adding t boxes not two in the same column.

Notation 1. If E is a complex vector bundle on a compact Kähler manifold X, then $\mu(E)$ will denote the slope of E, i.e. the degree of E divided by the rank of E.

3. The Proof

Theorem 2. *For any complex vector bundle E of rank e on a compact Kähler manifold X, we have that*

$$\mu(S^\lambda E) = |\lambda| \, \mu(E)$$

for any $\lambda = (\lambda_1, ..., \lambda_{e-1})$ with $\lambda_1 \geq ... \geq \lambda_{e-1} \geq 0$, where $|\lambda| = \lambda_1 + ... + \lambda_{e-1}$.

Proof. • First we prove the result in the case $\lambda = (m, 0, ..., 0)$, i.e. in the case of the symmetric powers of E. Obviously

$$\mu(S^m E) = \frac{\deg(S^m E)}{rk(S^m E)} = \frac{\deg(S^m E)}{\binom{e+m-1}{m}},$$

so we have to prove that

$$\deg(S^m E) = \binom{e+m-1}{e} \deg(E).$$

We prove it by induction on e. The case $e = 1$ is trivial. By the splitting principle, we can suppose that

$$E = E_1 \oplus ..... \oplus E_e,$$

where $rk(E_i) = 1$ for $i = 1, ..., e$. So

$$\begin{aligned}
\deg(S^m E) &= \deg(\oplus_{i_1,...,i_e \in \mathbb{N},\, i_1+....+i_e=m} S^{i_1}E_1 \otimes .... \otimes S^{i_e}E_e) \\
&= \sum_{i_1,...,i_e \in \mathbb{N},\, i_1+....+i_e=m} \deg(S^{i_1}E_1 \otimes .... \otimes S^{i_e}E_e) \\
&= \sum_{i_1,...,i_e \in \mathbb{N},\, i_1+....+i_e=m} \left[\deg(S^{i_1}E_1) + ... + \deg(S^{i_e}E_e)\right] \\
&= \sum_{i_1,...,i_e \in \mathbb{N},\, i_1+....+i_e=m} \left[i_1 \deg(E_1) + ... + i_e \deg(E_e)\right].
\end{aligned}$$

Observe that coefficients of $\deg(E_1)$, ..., $\deg(E_e)$ in the above formula must be equal. Besides the sum of the coefficients of $\deg(E_1)$, ..., $\deg(E_e)$ must be

$$\sum_{i_1,...,i_e \in \mathbb{N},\; i_1+....+i_e=m} i_1 + ... + i_e,$$

that is $\binom{e-1+m}{m} m$. Therefore the coefficient of $\deg(E_i)$ for any $i = 1, ..., e$ must be $\frac{m}{e}\binom{e-1+m}{m}$. Thus we get

$$\begin{aligned}
\deg(E) &= \frac{m}{e}\binom{e-1+m}{m}(\deg(E_1) + ... + \deg(E_e)) \\
&= \frac{m}{e}\binom{e-1+m}{m}\deg(E),
\end{aligned}$$

as we wanted to prove.

• Now we prove the result in general by induction on the number of the rows of the Young diagram of λ. If the number of the rows is 1, we already know the result.

So suppose that the statement holds for $S^\alpha E$ with the number of the rows of α less or equal than k.

We want to prove the statement when the number of the rows is less or equal than $k+1$; we show it by induction on the number t of the elements of the $(k+1)$-th row. If $t=0$ we know the statement by induction assumption. So suppose that the Young diagram of λ has $k+1$ rows and t elements in the $(k+1)$-th row.

We define ν to be the Young diagram we get from λ by deleting the last row.

Consider Pieri's formula applied to $S^\nu E \otimes S^t E$:

$$S^\nu E \otimes S^t E = \oplus_{\gamma \in \Gamma} S^\gamma E,$$

where Γ is the set of all the partitions $\gamma = (\gamma_1, \gamma_2, ...)$ with $\gamma_1 \geq \gamma_2 \geq ...$ of $|\nu| + t$ whose Young diagrams are obtained from the Young diagram of ν adding t boxes not two in the same column. Obviuosly $\lambda \in \Gamma$, so we can write $\Gamma = \{\lambda\} \cup \Gamma'$. We have:

$$\mu(S^\nu E \otimes S^t E) = \mu(\oplus_{\gamma \in \Gamma} S^\gamma E),$$

therefore

$$\mu(S^\nu E) + \mu(S^t E) = \mu(\oplus_{\gamma \in \Gamma'} S^\gamma E \oplus S^\lambda E),$$

then, by induction assumption,

$$|\nu|\mu(E) + t\mu(E) = \mu(\oplus_{\gamma \in \Gamma'} S^\gamma E \oplus S^\lambda E).$$

Hence we get:

$$(|\nu| + t)\mu(E) = \frac{\sum_{\gamma \in \Gamma'} \deg(S^\gamma E) + \deg(S^\lambda E)}{rk(S^\nu E \otimes S^t E)}.$$

Therefore

$$rk(S^\nu E \otimes S^t E)\ (|\nu| + t)\ \mu(E) = \sum_{\gamma \in \Gamma'} \deg(S^\gamma E) + \deg(S^\lambda E).$$

Hence

$$\begin{aligned}
\deg(S^\lambda E) &= rk(S^\nu E \otimes S^t E)\ (|\nu| + t)\ \mu(E) - \sum_{\gamma \in \Gamma'} \deg(S^\gamma E) \\
&= rk(S^\nu E \otimes S^t E)\ (|\nu| + t)\ \mu(E) - \sum_{\gamma \in \Gamma'} rk(S^\gamma E)\mu(S^\gamma E) \\
&= rk(S^\nu E \otimes S^t E)\ (|\nu| + t)\ \mu(E) - \sum_{\gamma \in \Gamma'} rk(S^\gamma E)\ |\gamma|\ \mu(E)
\end{aligned}$$

$$= \mu(E)\ (|\nu| + t)\ \left[rk(S^{\nu}E \otimes S^{t}E) - \sum_{\gamma \in \Gamma'} rk(S^{\gamma}E) \right] = \mu(E)\ (|\nu| + t)\ rk(S^{\lambda}E)$$
$$= rk(S^{\lambda}E)\ |\lambda|\ \mu(E),$$

where the last but three equality holds by induction assumption (induction on t) and the last equality and the last but two equality hold because $|\lambda| = |\gamma| = |\nu| + t$. □

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