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FUZZY CLOSURE SYSTEMS ON COMPLETE RESIDUATED LATTICES

Yong Chan Kim
Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: We investigate the properties of fuzzy closure systems on complete residuated lattices. In particular, we study the relations between fuzzy closure (interior) operators and fuzzy closure (interior) systems.

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1. Introduction

The notion of closure systems and closure operators facilitated to study topological structures, logic and lattices. Recently, Bělohlávek [2] investigate the properties of fuzzy closure systems and fuzzy closure operators on residual lattices which supports part of foundation of theoretic computer science. Guo et.al [6] introduced fuzzy closure systems in a sense as the least upper bound on fuzzy partial ordered sets. It is a generalization of Bělohlávek's fuzzy closure system.

In this paper, we investigate the properties of fuzzy closure system with respect to the least upper bound on a complete residuated lattice. In particular, we study the relations between fuzzy closure (open) operators and fuzzy closure (interior) systems.

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2. Preliminaries

Definition 1. (see [1-3], [6-9]) A structure $(L, \vee, \wedge, \odot, \to, \bot, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:

(R1) $(L, \vee, \wedge, \top, \bot)$ is a complete lattice with greatest element $\top$ and the least element $\bot$;

(R2) $(L, \odot, \top)$ is a commutative monoid;

(R3) it satisfies a residuation, i.e.

$$a \odot b \leq c \text{ iff } a \leq b \to c.$$

Remark 2. (see [6-9]) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ is a complete residuated lattice.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, \top, \bot)$ is a complete residuated lattice with the law of double negation defined as $a = (a^*)^*$ where $a^* = a \to \bot$.

Lemma 3. (see [8]) *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(2) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(3) $(x \odot y) \to z = x \to (y \to z)$.

(4) $x \leq (x \to y) \to y$ *and* $x \odot (x \to y) \leq y$.

(5) $(x \to y) \odot (y \to z) \leq x \to z$.

(6) $(x \to z) \leq (x \odot y) \to (z \odot y)$.

(8) $(x \to y) \leq (y \to z) \to (x \to z)$.

(9) $(x \to y) \leq (z \to x) \to (z \to y)$.

(10) $x \to y = \top$ *iff* $x \leq y$.

(11) $x \to y = y^* \to x^*$.

(12) $(x \to y)^* = x \odot y^*$.

(13) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

Definition 4. Let X be a set. A function $e_X : X \times X \to L$ is called a *fuzzy partial order* if it satisfies the following conditions :

(O1) $e_X(x, x) = \top$ for all $x \in X$, (O2) If $e_X(x,y) = e_X(y,x) = \top$, then $x = y$,

(O3) $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$, for all $x, y, z \in X$.
The pair (X, e_X) is called a *fuzzy partial ordered se.*

Example 5. (1) We define a function $e_L : L \times L \to L$ as

$$e_L(x,y) = x \to y.$$

By Lemma 3(5), (L, e_L) is a fuzzy partial ordered set.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A,B) = \bigwedge_{x \in X} (A(x) \to B(x)).$$

By Lemma 3(5), (L^X, e_{L^X}) is a fuzzy partial ordered set.

3. Fuzzy Closure Systems on Residuated Lattices

Definition 6. (see [6]) Let (X, e_X) be a fuzzy partial ordered set. A subset $M \subset X$ is called a *fuzzy closure system* if for any $x \in X$, there exists $m_x \in M$ such that

(M1) $e_X(x, m_x) = \top$ for all $x \in X$,

(M2) $e_X(n, m_x) \odot e_X(x, m) \leq e_X(n, m)$, for all $m, n \in M$.
We call m_x the *least upper bound* of x in M.

Remark 7. In the above definition, if m_x^1 and m_x are the least upper bounds of x, then , by (M2), $\top = e_X(m_x^1, m_x^1) \odot e_X(x, m_x) \leq e_X(m_x^1, m_x)$ and $\top = e_X(m_x, m_x) \odot e_X(x, m_x^1) \leq e_X(m_x, m_x^1)$;i.e. $e_X(m_x^1, m_x) = e_X(m_x, m_x^1) = \top$. Hence $m_x^1 = m_x$. Thus the least upper bound m_x is unique.

Theorem 8. *Let $\mathcal{M} = \{A_i \mid i \in \Gamma\}$ be a subset of L^X. The following statement are equivalent:*

(1) *$\mathcal{M} = \{A_i \mid i \in \Gamma\}$ is a fuzzy closure system on L^X.*

(2) *For any $A \in L^X$, there exists $m_A = \bigwedge_{i \in \Gamma}(e_{L^X}(A, A_i) \to A_i) \in \mathcal{M}$.*

(3)*$\bigwedge_{i \in \Gamma}(a_i \to A_i) \in \mathcal{M}$.*

Proof. (1)$\Rightarrow$ (2). For each $A \in L^X$ and $m_A \in \mathcal{M}$, we will show that $m_A = \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)$. Since $m_A \in \mathcal{M}$ and $e_{L^X}(A, m_A) = \top$, we have

$$\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i) \leq e_{L^X}(A, m_A) \to m_A(x) = m_A(x).$$

Since $e_{L^X}(A, A_i) \odot e_{L^X}(A_j, m_A) \leq e_{L^X}(A_j, A_i)$ from (M2) and $m_A \in \mathcal{M}$, put $A_j = m_A$, then

$$\begin{aligned} e_{L^X}(A, A_i) &= e_{L^X}(A, A_i) \odot e_{L^X}(m_A, m_A) \\ &\leq e_{L^X}(m_A, A_i) \leq m_A(x) \to A_i(x). \end{aligned}$$

Then $m_A(x) \leq e_{L^X}(m_A, A_i) \to A_i(x)$. Hence $m_A = \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)$.

(2)$\Rightarrow$ (1). For any $A \in L^X$, put $m_A = \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)$.

$$\begin{aligned} e_{L^X}(A, m_A) &= e_{L^X}(A, \textstyle\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)) \\ &= \textstyle\bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i(x))\Big) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{i\in\Gamma}\Big(A(x) \to (e_{L^X}(A, A_i) \to A_i(x))\Big) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{i\in\Gamma}\Big(e_{L^X}(A, A_i) \to (A(x) \to A_i(x))\Big) \\ &= \textstyle\bigwedge_{i\in\Gamma}\Big(e_{L^X}(A, A_i) \to \bigwedge_{x\in X}(A(x) \to A_i(x))\Big) \\ &= \top. \end{aligned}$$

$$\begin{aligned} e_{L^X}(A, A_i) \odot m_A(x) &= e_{L^X}(A, A_i) \odot \textstyle\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)(x)) \\ &\leq e_{L^X}(A, A_i) \odot (e_{L^X}(A, A_i) \to A_i(x)) \leq A_i(x). \end{aligned}$$

Hence $e_{L^X}(A, A_i) \leq m_A(x) \to A_i(x)$. Thus, $e_{L^X}(A, A_i) \leq e_{L^X}(m_A, A)$.

Therefore

$$e_{L^X}(A, A_i) \odot e_{L^X}(A_j, m_A) \leq e_{L^X}(m_A, A_i) \odot e_{L^X}(A_j, m_A) \leq e_{L^X}(A_j, A_i).$$

(2) $\Rightarrow$ (3) Put $A = \bigwedge_{i\in\Gamma}(a_i \to A_i)$. Then $A \leq e_{L^X}(A, A_i) \to A_i$ from Lemma 3(4). Thus

$$A \leq \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i).$$

Since $\bigwedge_{i\in\Gamma}(a_i \to A_i) \odot a_i \leq (a_i \to A_i) \odot a_i \leq A_i$, then

$$e_{L^X}(A, A_i) = e_{L^X}(\bigwedge_{i\in\Gamma}(a_i \to A_i), A_i) \geq a_i.$$

Thus

$$(e_{L^X}(A, A_i) \to A_i) \odot a_i \le (e_{L^X}(A, A_i) \to A_i) \odot e_{L^X}(A, A_i) \le A_i.$$

Hence $e_{L^X}(A, A_i) \to A_i \le a_i \to A_i$. Hence $\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i) \le A$. It follows $\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i) = A \in \mathcal{M}$.

(3) $\Rightarrow$(2) Put $a_i = e_{L^X}(A, A_i)$, it easily proved. □

Remark 9. The condition in Theorem 8(2) is the fuzzy closure system defined by Bělohlávek in [2].

Definition 10. (see [6]) Let (X, e_X) be a fuzzy partial ordered set. A subset $K \subset X$ is called a *fuzzy interior system* if for any $x \in X$, there exists $k_x \in K$ such that:

(K1) $e_X(k_x, x) = \top$ for all $x \in X$,

(K2) $e_X(m, x) \odot e_X(k_x, n) \le e_X(m, n)$, for all $m, n \in K$.

We call k_x the *greatest lower bound* of x in K.

Remark 11. In the above definition, by a similar method as Remark 7, the greatest lower bound k_x is unique.

Theorem 12. *Let $\mathcal{K} = \{A_i \mid i \in \Gamma\}$ be a subset of L^X. The following statement are equivalent:*

(1) *$\mathcal{K}$ is a left interior system on L^X.*

(2) *For any $A \in L^X$, $\bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i) \in \mathcal{K}$.*

(3) *For any $a_i \in L$, $\bigvee_{i\in\Gamma}(a_i \odot A_i) \in \mathcal{K}$.*

Proof. (1)$\Rightarrow$ (2). We show that $k_A = \bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i)$. Since $k_A \in \mathcal{M}$,

$$\bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i(x)) \ge e_{L^X}(k_A, A) \odot k_A(x) = k_A(x).$$

By (K2), since $e_{L^X}(A_i, A) \odot e_{L^X}(k_A, A_j) \le e_{L^X}(A_i, A_j)$ and $k_A \in \mathcal{K}$, put $A_j = k_A$, then

$$e_{L^X}(A_i, A) = e_{L^X}(k_A, k_A) \odot e_{L^X}(A_i, A) \le e_{L^X}(A_i, k_A).$$

Hence $e_{L^X}(A_i, A) \le A_i(x) \to k_A(x)$. Thus, $\bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i(x)) \le k_A(x)$. Therefore $k_A = \bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i)$.

(2)⇒ (1). For any $A \in L^X$, put $k_A = \bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i)$.

$$\begin{aligned} e_{L^X}(k_A, A) &= e_{L^X}(\textstyle\bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i \to A)) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{i\in\Gamma}\Big(e_{L^X}(A_i, A) \to (A_i(x) \to A(x))\Big) \\ &= \textstyle\bigwedge_{i\in\Gamma}\Big(e_{L^X}(A_i, A) \to \bigwedge_{x\in X}(A_i(x) \to A(x))\Big) \\ &= \textstyle\bigwedge_{i\in\Gamma}\Big(e_{L^X}(A_i, A) \to e_{L^X}(A_i, A)\Big) = \top. \end{aligned}$$

Since $e_{L^X}(A_i, e_{L^X}(A, A_i) \odot A_i) \geq e_{L^X}(A_i, A)$, we have

$$\begin{aligned} e_{L^X}(A_i, k_A) &= e_{L^X}(A_i, \textstyle\bigvee_{i\in\Gamma}(e_{L^X}(A_i, A) \odot A_i) \\ &\geq e_{L^X}(A_i, e_{L^X}(A, A_i) \odot A_i) \geq e_{L^X}(A_i, A). \end{aligned}$$

Thus $e_{L^X}(A_i, A) \odot e_{L^X}(k_A, A_j) \leq e_{L^X}(A_i, k_A) \odot e_{L^X}(k_A, A_j) \leq e_{L^X}(A_i, A_j)$.

(2) ⇒ (3) Put $A = \bigvee_{i\in\Gamma}(b_i \odot B_i)$. Then $A \geq e_{L^X}(B_i, A) \odot B_i$. Thus $A \geq \bigvee_{i\in\Gamma}(e_{L^X}(B_i, A) \odot B_i)$. On the other hand, since $e_{L^X}(B_i, A) = e_{L^X}(B_i, \bigvee_{i\in\Gamma}(b_i \odot B_i)) \geq b_i$, then $b_i \odot B_i \leq \bigvee_{i\in\Gamma}(e_{L^X}(B_i, A) \odot B_i)$. Hence $A \leq \bigvee_{i\in\Gamma}(e_{L^X}(B_i, A) \odot B_i)$. It follows $\bigvee_{i\in\Gamma}(e_{L^X}(B_i, A) \odot B_i) = A \in \mathcal{K}$.

(3)⇒(2) Put $b_i = e_{L^X}(B_i, A)$, it easily proved. □

Theorem 13. *Let (X, e_X) be a fuzzy partial ordered set.*

(1) If (X, M) is a fuzzy closure system on X, then (X, M) is a fuzzy interior system on (X, e_X^{-1}) where $e_X^{-1}(x, y) = e_X(y, x)$.

(2) If (X, K) is a fuzzy interior system on X, then (X, K) is a fuzzy closure system on (X, e_X^{-1}).

Proof. (1) e_X^{-1} is a fuzzy partial order from:

$$\begin{aligned} &e_X^{-1}(x, y) \odot e_X^{-1}(y, z) = e_X(y, x) \odot e_X(z, y) \\ &\leq e_X(z, x) = e_X^{-1}(x, z). \end{aligned}$$

Since $e_X(x, m_x) = \top$ and $e_X(x, m) \odot e_X(n, m_x) \leq e_X(n, m)$, for all $m, n \in M$, we have

$$\begin{aligned} &e_X^{-1}(m_x, x) = e_X(x, m_x) = \top, \\ &e_X^{-1}(m, x) \odot e_X^{-1}(m_x, n) = e_X(x, m) \odot e_X(n, m_x) \\ &\leq e_X(n, m) = e_X^{-1}(m, n). \end{aligned}$$

Hence $m_x \in M$ is the greatest upper bound of x. Thus M is a fuzzy interior system on (X, e_X^{-1})

(2) It is similarly proved. □

Theorem 14. *Let (L^X, e_{L^X}) be a fuzzy partial ordered set.*

(1) If $\mathcal{M}$ is a fuzzy closure system on L^X, then $\mathcal{M}$ is a fuzzy interior system on $(L^X, e^{-1}_{L^X})$ where

$$e^{-1}_{L^X}(A, B) = e_{L^X}(B, A) = e_{L^X}(A^*, B^*).$$

In particular, put $\mathcal{K}_M = \{B^* \mid B \in \mathcal{M}\}$. Then $k_D = (m_{D^*})^*$ is the greatest upper bound of D on the fuzzy interior system $\mathcal{K}_M$.

(2) If $\mathcal{K}$ is a fuzzy interior system on L^X, then $\mathcal{K}$ is a fuzzy closure system on $(L^X, e^{-1}_{L^X})$. In particular, put $\mathcal{M}_K = \{B^* \mid B \in \mathcal{K}\}$. Then $m_A = (k_{A^*})^*$ is the least lower bound of A on the fuzzy closure system $\mathcal{M}_K$.

Proof. (1) $e^{-1}_{L^X}(A, B) = e_{L^X}(B, A) = e_{L^X}(A^*, B^*)$ from:

$$\begin{aligned} e^{-1}_{L^X}(A, B) &= e_{L^X}(B, A) = \bigwedge_{x \in X}(B(x) \to A(x)) \\ &= \bigwedge_{x \in X}(A^*(x) \to B^*(x)) = e_{L^X}(A^*, B^*). \end{aligned}$$

Since $e_{L^X}((m_A)^*, A^*) = e_{L^X}(A, m_A) = \top$ and $e_{L^X}(A, B) \odot e_{L^X}(C, m_A) \le e_{L^X}(C, B)$, for all $B, C \in M$, we have

$$\begin{aligned} &e_{L^X}((m_A)^*, A^*) = e^{-1}_{L^X}(m_A, A) = e_{L^X}(A, m_A) = \top, \\ &e_{L^X}(B^*, A^*) \odot e_{L^X}((m_A)^*, C^*) \\ &= e^{-1}_{L^X}(B, A) \odot e^{-1}_{L^X}(m_A, C) = e_{L^X}(A, B) \odot e_{L^X}(C, m_A) \\ &\le e_{L^X}(C, B) = e^{-1}_{L^X}(B, C) = e_{L^X}(B^*, C^*). \end{aligned}$$

Put $A = D^*$. Since $A^* = (D^*)^* = D$, we have

$$\begin{aligned} &e_{L^X}((m_{D^*})^*, D) = \top, \\ &e_{L^X}(B^*, D) \odot e_{L^X}((m_{D^*})^*, C^*) \le e_{L^X}(B^*, C^*). \end{aligned}$$

Put $K_M = \{B^* \mid B \in \mathcal{M}\}$. Then $k_D = (m_{D^*})^*$ is the greatest upper bound of D on a fuzzy interior system (X, K_M).

(2) Since $e_{L^X}(A^*, (k_A)^*) = e_{L^X}(k_A, A) = \top$ and $e_{L^X}(B, A) \odot e_{L^X}(k_A, C) \le e_X(B, C)$, for all $B, C \in K$, we have

$$\begin{aligned} &e_{L^X}(A^*, (k_A)^*) = e^{-1}_{L^X}(A, k_A) = e_{L^X}(k_A, A) = \top, \\ &e_{L^X}(A^*, B^*) \odot e_{L^X}(C^*, (k_A)^*) \\ &= e^{-1}_{L^X}(A, B) \odot e^{-1}_{L^X}(C, k_A) = e_{L^X}(B, A) \odot e_{L^X}(k_A, C) \\ &\le e_{L^X}(B, C) = e^{-1}_{L^X}(C, B) = e_{L^X}(B^*, C^*). \end{aligned}$$

Put $A = D^*$. Since $A^* = (D^*)^* = D$, we have

$$\begin{array}{l} e_{L^X}(D, (k_{D^*})^*) = \top, \\ e_{L^X}(D, B^*) \odot e_{L^X}(C^*, (k_{D^*})^*) \le e_{L^X}(B^*, C^*). \end{array}$$

Put $\mathcal{M}_K = \{B^* \mid B \in \mathcal{K}\}$. Then $m_D = (k_{D^*})^*$ is the least lower bound of D on a fuzzy closure system $(X, \mathcal{M}_K)$. □

Definition 15. (see [6, 9]) Let (X, e_X) be a fuzzy partial ordered set. An operator $C : X \to X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:

(C1) $e_X(x, C(x)) = e_X(C(C(x)), C(x)) = \top$,

(C2) $e_X(x, y) \le e_X(C(x), C(y))$.
The pair (X, C) is called a *fuzzy closure space.*

Definition 16. (see [6, 9] Let (X, e_X) be a fuzzy partial ordered set. An operator $I : X \to X$ is called a *fuzzy interior operator* on X if it satisfies the following conditions:

(I1) $e_X(I(x), x) = e_X(I(x), I(I(x))) = \top$,

(I2) $e_X(x, y) \le e_X(I(x), I(y))$.

The pair (X, I) is called a *fuzzy interior space.*

Theorem 17. *Let (X, e_X) be a fuzzy partial ordered set.*

(1) Let (X, I) be a fuzzy interior operator. Define $K_I = \{I(x) \mid x \in X\}$ and $K_I = \{I(x) \mid x \in X\}$. Then (X, K_I) is a fuzzy interior system.

(2) Let (X, K) be a fuzzy interior system. Define $I_K(x) = k_x$ where $I(x)$ is the greatest lower bound of x. Then (X, I_K) is a fuzzy interior operator.

Proof. (1) Let I be a fuzzy interior operator. Since $I(x) \in K_I$, then $e_X(I(x), x) = \top$. For $m \in K_I$, there exists $x_m \in X$ such that $I(x_m) = m$. Since I is a fuzzy interior operator, $\top = e_X(m, x_m) \le e_X(I(m), I(x_m)) = e_X(I(m), m)$. Moreover, $e_X(I(x_m), I(I(x_m))) = e_X(m, I(m)) = \top$. Since

$$e_X(I(m), m) = e_X(m, I(m)) = \top,$$

$I(m) = m$. For any $m, n \in K_I$, we have $I(m) = m$ and $I(m) = m$. Thus

$$\begin{array}{rl} e_X(m, x) \odot e_X(I(x), n) & \le e_X(I(m), I(x)) \odot e_X(I(x), n) \\ & \le e_X(I(m), n) = e_X(m, n). \end{array}$$

Hence $I(x)$ is the greatest lower bound of x. Thus K_I be a fuzzy interior system.

(2) Let K be a fuzzy interior system. Since $I_K(x) = k_x$, $e_X = (I_K(x), x) = e_X(k_x, x) = \top$. For each $x_1, x_2 \in X$,

$$e_X(x_1, x_2) = e_X(I_K(x_1), x_1) \odot e_X(x_1, x_2) \le e_X(I_K(x_1), x_2).$$

By (K2), since $e_X(m, x) \odot e_X(k_x, n) \le e_X(m, n)$, for all $m, n \in K$, put $m = I_K(x_1)$, $x = x_2$ and $n = I_K(x_2)$, then $k_x = I_K(x_2)$ and

$$e_X(I_K(x_1), x_2) \odot e_X(I_K(x_2), I_K(x_2)) \le e_X(I_K(x_1), I_K(x_2)).$$

Hence $e_X(x_1, x_2) \le e_X(I_K(x_1), I_K(x_2))$. By (K2), $e_X(m, x) \odot e_X(k_x, n) \le e_X(m, n)$, for all $m, n \in K$. Since $I_K(x) \in K$, put $m = x = I_K(x)$ and $n = I_K(I_K(x))$, then

$$\top = e_X(I_K(x), I_K(x)) \odot e_X(I_K(I_K(x)), I_K(I_K(x))) \le e_X(I_K(x), I_K(I_K(x))).$$

Hence $e_X(I_K(x), I_K(I_K(x))) = \top$. Thus I_K is a fuzzy interior operator. □

Theorem 18. *Let (L^X, e_{L^X}) be a fuzzy partial ordered set.*

(1) *Let (L^X, I) be a fuzzy interior operator. Define $K_I = \{I(A) \mid A \in L^X\}$. Then (L^X, K_I) is a fuzzy interior system.*

(2) *Let (L^X, K) is a fuzzy interior system. Define $I_K(A) = k_A$. Then (L^X, I_K) is a fuzzy interior operator.*

(3) *In (1) and (2), $I_{K_I} = I$ and $K_{I_K} = K$.*

Proof. (1) and (2) follow from Theorem 17.

(3) Since $K_I = \{I(A) \mid A \in L^X\}$, by Theorems 12(2) and 17(1),

$$I_{K_I}(B) = \bigvee_{A \in L^X} (I(A) \odot e_{L^X}(I(A), B))$$

Since $e_{L^X}(I(B), B) = \top$,

$$I_{K_I}(B) \ge I(B) \odot e_{L^X}(I(B), B) = I(B).$$

Since $I(A) \odot e_{L^X}(I(A), B) \le B$, then

$$B \ge I_{K_I}(B) = \bigvee_{A \in L^X} (I(A) \odot e_{L^X}(I(A), B)) \in K_I,$$

by (K2),

$$\top = e_{L^X}(I(B), I(B)) \odot e_X(I_{K_I}(B), B) \le e_X(I_{K_I}(B), I(B)).$$

Hence $I_{K_I}(B) \le I(B)$. Thus, $I_{K_I}(B) = I(B)$.

Let $K = \{A_i \mid i \in \Gamma\}$ be a fuzzy interior system. Then $I_K(A) = \bigvee_{i\in\Gamma}(A_i \odot e_{L^X}(A, A_i))$. Hence $K_{I_K} = \{\bigvee_{i\in\Gamma}(A_i \odot e_{L^X}(A_i, A)) \mid A \in L^X\}$. Let $A_j \in K$. Then $\bigvee_{i\in\Gamma}(A_i \odot e_{L^X}(A_i, A_j)) \ge A_j \odot e_{L^X}(A_j, A_j) = A_j$. Since $A_i \odot e_{L^X}(A_i, A_j) \le A_j$, then $\bigvee_{i\in\Gamma}(A_i \odot e_{L^X}(A_i, A_j)) \le A_j$. Thus

$$\bigvee_{i\in\Gamma}(A_i \odot e_{L^X}(A_i, A_j)) \ge A_j \odot e_{L^X}(A_j, A_j) = A_j \in K_{I_K}.$$

Let $B \in K_{I_K}$. By Theorem 12(2), $B \in K$. Hence $K_{I_K} = K$. □

Corollary 19. *Let (X, e_X) be a fuzzy partial ordered set.*

(1) *Let (X, C) be a fuzzy closure operator. Define $M_C = \{C(x) \mid x \in X\}$. Then (X, M_C) is a fuzzy closure system.*

(2) *Let (X, M) be a fuzzy closure system. Define $C_M(x) = m_x$. Then (X, C_M) is a fuzzy closure operator.*

Theorem 20. *Let (L^X, e_{L^X}) be a fuzzy partial ordered set.*

(1) *Let (L^X, C) be a fuzzy closure operator. Define $M_C = \{C(A) \mid A \in L^X\}$. Then (L^X, M_C) is a fuzzy closure system.*

(2) *Let (L^X, M) be a fuzzy closure system. Define $C_M(A) = m_A$. Then (L^X, C_M) is a fuzzy closure operator.*

(3) *In (1) and (2), $C_{M_C} = C$ and $M_{C_M} = M$.*

Proof. (3) Since $M_C = \{C(A) \mid A \in L^X\}$, by Theorem 8(2),

$$C_{M_C}(B) = \bigwedge_{A\in L^X}(e_{L^X}(B, C(A)) \to C(A)$$

Since $e_{L^X}(B, C(B)) = \top$,

$$C_{M_C}(B) \le e_{L^X}(B, C(B)) \to C(B) = C(B).$$

Since $B \odot e_{L^X}(B, C(A)) \le C(A)$, then

$$B \le \bigwedge_{A\in L^X}(e_{L^X}(B, C(A)) \to C(A)$$

Since $C_{M_C}(B) = \bigwedge_{A\in L^X}(e_{L^X}(B, C(A)) \to C(A) \in M_C$, by (M2),

$$\top = e_{L^X}(C(B), C(B)) \odot e_X(B, C_{M_C}(B)) \le e_X(C(B), C_{M_C}(B)).$$

Hence $C(B) \le C_{M_C}(B)$. Thus, $C_{M_C} = C$.

Let $M = \{A_i \mid i \in \Gamma\}$ be a fuzzy closure system. Then $C_M(A) = \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i)$. Hence $M_{C_M} = \{\bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i) \mid A \in L^X\}$. Let $A_j \in M$. Then $\bigwedge_{i\in\Gamma}(e_{L^X}(A_j, A_i) \to A_i) = e_{L^X}(A_j, A_j) \to A_j = A_j$. Let $k_A = \bigwedge_{i\in\Gamma}(e_{L^X}(A, A_i) \to A_i) \in M_{C_M}$. By Theorem 8(2), $k_A \in M$. □

Theorem 21. *Let (X, e_X) be a fuzzy partial ordered set.*

(1) If (X, C) be a fuzzy closure operator, then (X, C) is a fuzzy interior operator on (X, e_X^{-1}) where $e_X^{-1}(x, y) = e_X(y, x)$.

(2) If (X, I) be a fuzzy interior operator, then (X, I) is a fuzzy closure operator on (X, e_X^{-1}).

Theorem 22. *Let (L^X, e_{L^X}) be a fuzzy partial ordered set.*

(1) If (L^X, C) is a fuzzy closure operator on L^X, then (L^X, C) is a fuzzy interior system on $(L^X, e_{L^X}^{-1})$ where

$$e_{L^X}^{-1}(A, B) = e_{L^X}(B, A) = e_{L^X}(A^*, B^*).$$

In particular, define $I(A) = C(A^*)^*$ and $I(A) = C(A^*)^*$. Then (X, I) is a fuzzy interior operator on (L^X, e_{L^X}).

(2) If (X, I) is a fuzzy interior operator on L^X, then (X, I) is a fuzzy closure operator on $(L^X, e_{L^X}^{-1})$.

In particular, define $C(A) = I(A^*)^*$. Then (X, C) is a fuzzy closure operator on (L^X, e_{L^X}).

Proof. (1) Since $e_{L^X}^{-1}(A, B) = e_{L^X}(B, A) = e_{L^X}(A^*, B^*)$,

$$\begin{aligned}
e_{L^X}(C(A)^*, A^*) &= e_{L^X}^{-1}(C(A), A) = e_{L^X}(A, C(A)) = \top\\
e_{L^X}(C(A)^*, C(C(A))^*) &= e_{L^X}^{-1}(C(A), C(C(A)))\\
&= e_{L^X}(C(C(A)), C(A)) = \top\\
e_{L^X}(A^*, B^*) &= e_{L^X}^{-1}(A, B) = e_{L^X}(B, A)\\
&\le e_{L^X}(C(B), C(A)) = e_{L^X}(C(A)^*, C(B)^*).
\end{aligned}$$

Put $A = A^*$. Then

$$\begin{aligned}
e_{L^X}(C(A^*)^*, A) &= \top\\
e_{L^X}(C(A^*)^*, C(C(A^*))^*) &= \top\\
e_{L^X}(A, B) &\le e_{L^X}(C(A^*)^*, C(B^*)^*).
\end{aligned}$$

Put $I(A) = C(A^*)^*$. Then $I(I(A)) = C(C(A^*))^*(A)$. Thus

$$\begin{aligned} e_{L^X}(I(A), A) &= e_{L^X}(I(A), I(I(A))) = \top \\ e_{L^X}(A, B) &\leq e_{L^X}(I(A), I(A)). \end{aligned}$$

(2) Since $e^{-1}_{L^X}(A, B) = e_{L^X}(B, A) = e_{L^X}(A^*, B^*)$,

$$\begin{aligned} e_{L^X}(A^*, I(A)^*) &= e^{-1}_{L^X}(A, I(A)) = e_{L^X}(I(A), A) = \top \\ e_{L^X}(I(I(A))^*, I(A)^*) &= e^{-1}_{L^X}(I(I(A)), I(A)) \\ &= e_{L^X}(I(A), I(I(A))) = \top \\ e_{L^X}(A^*, B^*) &= e^{-1}_{L^X}(A, B) = e_{L^X}(B, A) \\ &\leq e_{L^X}(I(B), I(A)) = e_{L^X}(I(A)^*, I(B)^*). \end{aligned}$$

Put $A = A^*$. Then

$$\begin{aligned} e_{L^X}(A, I(A^*)^*) &= \top \\ e_{L^X}(I(A^*)^*, I(I(A^*))^*) &= \top \\ e_{L^X}(A, B) &\leq e_{L^X}(I(A^*)^*, I(B^*)^*). \end{aligned}$$

Put $C(A) = I(A^*)^*$. Then $C(C(A)) = I(I(A^*))^*$. Thus

$$\begin{aligned} e_{L^X}(A, C(A)) &= e_{L^X}(C(C(A)), C(A)) = \top \\ e_{L^X}(A, B) &\leq e_{L^X}(C(A), C(B)). \end{aligned}$$

□

Theorem 23. *The following statement are equivalent.*

(1) *An operator* $I : L^X \to L^X$ *is a fuzzy interior operator on* L^X

(2) I *satisfies* (I1) *and*
(II) $\alpha \odot I(A) \leq I(\alpha \odot A)$ *and* $I(A) \leq I(B)$ *for* $A \leq B$.

(3) I *satisfies* (I1) *and*
(III) $I(\alpha \to A) \leq \alpha \to I(A)$ *and* $I(A) \leq I(B)$ *for* $A \leq B$.

Proof. (1) ⇒ (2) If $A \leq B$, then $\top = e_{L^X}(A, B) \leq e_{L^X}(I(A), I(B))$. Thus, $I(A) \leq I(B)$. Since $\alpha \leq e_{L^X}(A, \alpha \odot A)$, we have $\alpha \leq e_{L^X}(A, \alpha \odot A) \leq e_{L^X}(I(A), I(\alpha \odot A))$. Thus $\alpha \odot I(A) \leq I(\alpha \odot A)$.

(2) ⇒ (1). Put $\alpha = e_{L^X}(A, B)$. By (II), $\alpha \odot I(A) \leq I(\alpha \odot A) = I(e_{L^X}(A, B) \odot A) \leq I(B)$. Hence $e_{L^X}(A, B) \leq e_{L^X}(I(A), I(B))$.

(2) ⇒ (3) Since $\alpha \odot I(\alpha \to A) \leq I(\alpha \odot (\alpha \to A)) \leq I(A)$, then $I(\alpha \to A) \leq \alpha \to I(A)$.

(3) $\Rightarrow$ (2) Since $I(\alpha \to \alpha \odot A) \le \alpha \to I(\alpha \odot A)$ iff $\alpha \odot I(\alpha \to \alpha \odot A) \le I(\alpha \odot A)$, we have

$$\alpha \odot I(A) \le \alpha \odot I(\alpha \to \alpha \odot A) \le I(\alpha \odot A).$$

□

Theorem 24. *The following statement are equivalent.*

(1) *An operator $C : L^X \to L^X$ is a fuzzy closure operator on L^X*

(2) *C satisfies* (C1) *and*
(II) *$\alpha \odot C(A) \le C(\alpha \odot A)$ and $C(A) \le C(B)$ for $A \le B$.*

(3) *C satisfies* (C1) *and*
(III) *$C(\alpha \to A) \le \alpha \to C(A)$ and $C(A) \le C(B)$ for $A \le B$.*

Proof. (1) $\Rightarrow$ (3). (C1) If $A \le B$, then $\top = e_{L^X}(A, B) \le e_{L^X}(C(A), C(B))$. Thus, $C(A) \le C(B)$.

(III) Since $\alpha \le e_{L^X}(\alpha \to A, A)$, we have $\alpha \le e_{L^X}(\alpha \to A, A) \le e_{L^X}(C(\alpha \to A), C(A))$. It implies $\alpha \odot C(\alpha \to A) \le C(A)$. Thus $C(\alpha \to A) \le \alpha \to C(A)$.

(3) $\Rightarrow$ (1). Put $\alpha = e_{L^X}(A, B)$. By (III), $C(e_{L^X}(A, B) \to B) \le e_{L^X}(A, B) \to C(B)$ implies $e_{L^X}(A, B) \le C(e_{L^X}(A, B) \to B) \to C(B)$. Since $e_{L^X}(A, B) \odot A \le B$, then $A \le e_{L^X}(A, B) \to B$. Since $C(A) \le C(e_{L^X}(A, B) \to B)$, we have

$$e_{L^X}(A, B) \le C(e_{L^X}(A, B) \to B) \to C(B) \le C(A) \to C(B).$$

Thus $e_{L^X}(A, B) \le e_{L^X}(C(A), C(B))$.

(3) $\Rightarrow$ (2) Let C be an operator satisfying (3). Since $C(\alpha \to \alpha \odot A) \le \alpha \to C(\alpha \odot A)$ iff $\alpha \odot C(\alpha \to \alpha \odot A) \le C(\alpha \odot A)$, we have

$$\alpha \odot C(A) \le \alpha \odot C(\alpha \to \alpha \odot A) \le C(\alpha \odot A).$$

(2) $\Rightarrow$ (3) $\alpha \odot C(\alpha \to A) \le C(\alpha \odot (\alpha \to A)) \le C(A)$, then $C(\alpha \to A) \le \alpha \to C(A)$. □

Theorem 25. (1) *If $\mathcal{K} = \{A_i \mid i \in \Gamma\}$ is a fuzzy interior system on L^X, we define $I_{\mathcal{K}} : L^X \to L^X$ as $I_{\mathcal{K}}(A) = \bigvee\{A_i \mid A_i \le A,\ A_i \in \mathcal{K}\}$. Then $I_{\mathcal{K}}(A) = \bigvee_{A_i \in \mathcal{K}}(e_{L^X}(A_i, A) \odot A_i)$ is a fuzzy interior operator on X.*

(2) *If $\mathcal{M} = \{A_i \mid i \in \Gamma\}$ is a fuzzy closure system on L^X, we define $C_{\mathcal{M}} : L^X \to L^X$ as $C_{\mathcal{M}}(A) = \bigwedge\{A_i \mid A \le A_i,\ A_i \in \mathcal{M}\}$. Then $C_{\mathcal{M}}(A) = \bigwedge_{A_i \in \mathcal{F}}(e_{L^X}(A, A_i) \to A_i)$ is a fuzzy closure operator on L^X.*

Proof. (1) Since $\bigvee_{i\in\Gamma}(e_{L^X}(A_i,A)\odot A_i)\le A$ and $\bigvee_{i\in\Gamma} e_{L^X}(A_i,A)\odot A_i\in\mathcal{K}$, we have $\bigvee_{i\in\Gamma}(e_{L^X}(A_i,A)\odot A_i)\le I_{\mathcal{K}}(A)$. Since $A_i\le A$, $e_{L^X}(A_i,A)\odot A_i=A_i$. Hence $I_{\mathcal{K}}(A)\le\bigvee_{i\in\Gamma}(e_{L^X}(A_i,A)\odot A_i)$. Thus, $I_{\mathcal{K}}(A)=\bigvee_{i\in\Gamma}(e_{L^X}(A_i,A)\odot A_i)$. By Theorem 18(2), $I_{\mathcal{K}}(A)=k_A$ is a fuzzy interior operator.

(2) Since $A\le\bigwedge_{i\in\Gamma}(e_{L^X}(A,A_i)\to A_i)=m_A\in\mathcal{M}$, we have $C_{\mathcal{M}}(A)\le\bigwedge_{i\in\Gamma}(e_{L^X}(A,A_i)\to A_i)$. For $A\le A_i$ with $A_i\in\mathcal{M}$, then $e_{L^X}(A,A_i)\to A_i=\top\to A_i=A_i$. Hence $\bigwedge_{A_i\in\mathcal{M}}(e_{L^X}(A,A_i)\to A_i)\le C_{\mathcal{M}}(A)$. By Theorem 20(2), $C_{\mathcal{M}}(A)=\bigwedge_{A_i\in\mathcal{M}}(e_{L^X}(A,A_i)\to A_i)$. □

Example 26. Let $(L=[0,1],\odot)$ be a residuated lattice defined by

$$x\odot y=(x+y-1)\vee 0,\quad x\to y=(1-x+y)\wedge 1$$

Let $X=\{a,b\}$ and $A\in L^X$ as follows: $A(a)=0.6, A(b)=0.8$.

Let $\mathcal{M}=\{\alpha\to A\mid\alpha\in L\}$ and $\mathcal{K}=\{\alpha\odot A\mid\alpha\in L\}$ be given.

(1) $\mathcal{M}$ is a fuzzy closure system from the following statements. For each $D\in L^X$,

$$\begin{aligned}C_{\mathcal{M}}(D)=m_D &=\textstyle\bigwedge_{\alpha\in L}(e_{L^X}(D,\alpha\to A)\to(\alpha\to A)\\ &=\textstyle\bigwedge_{\alpha\in L}((\alpha\to e_{L^X}(D,A))\to(\alpha\to A)\\ &\ge e_{L^X}(D,A)\to A,\\ C_{\mathcal{M}}(D)=m_D &=\textstyle\bigwedge_{\alpha\in L}(e_{L^X}(D,\alpha\to A)\to(\alpha\to A)\\ &\le(\top\to e_{L^X}(D,A))\to(\top\to A)\\ &\le e_{L^X}(D,A)\to A.\end{aligned}$$

Hence $C_{\mathcal{M}}(D)=m_D=e_{L^X}(D,A)\to A\in\mathcal{M}$. Moreover,

$$M_{C_{\mathcal{M}}}=\{e_{L^X}(D,A)\to A\mid D\in L^X\}=\{\alpha\to A\mid\alpha\in L\}=\mathcal{M}.$$

From Theorem 14(1), $\mathcal{K}_M=\{\alpha\odot A^*\mid\alpha\in L\}$ is a fuzzy interior system where $A^*(a)=0.4, A^*(b)=0.2$. Moreover,

$$\begin{aligned}I_{\mathcal{K}_M}(D)=k_D &=(m_{D^*})^*=(e_{L^X}(A,D)\to A)^*\\ &=e_{L^X}(A^*,D)\odot A^*.\end{aligned}$$

(2) $\mathcal{K}$ is a fuzzy interior system from the following statements. For each $D\in L^X$,

$$\begin{aligned}I_{\mathcal{K}}(D)=k_D &=\textstyle\bigvee_{\alpha\in L}(e_{L^X}(\alpha\odot A,D)\odot(\alpha\odot A)\\ &=\textstyle\bigwedge_{\alpha\in L}((\alpha\to e_{L^X}(A,D))\odot\alpha)\odot A\\ &\le e_{L^X}(A,D)\odot A,\\ I_{\mathcal{K}}(D)=k_D &=\textstyle\bigvee_{\alpha\in L}(e_{L^X}(\alpha\odot A,D)\odot(\alpha\odot A))\\ &\ge e_{L^X}(\top\odot A,D)\odot(\top\odot A)\\ &=e_{L^X}(A,D)\odot A.\end{aligned}$$

Hence $I_{\mathcal{K}}(D) = k_D = e_{L^X}(A, D) \odot A \in \mathcal{K}$. Moreover,

$$K_{I_{\mathcal{K}}} = \{e_{L^X}(A, D) \odot A \mid D \in L^X\} = \{\alpha \odot A \mid \alpha \in L\} = \mathcal{K}.$$

From Theorem 14(2), $\mathcal{M}_K = \{\alpha \to A^* \mid \alpha \in L\}$ is a fuzzy closure system where $A^*(a) = 0.4, A^*(b) = 0.2$. Moreover,

$$\begin{aligned} C_{\mathcal{M}_K}(D) = m_D &= (k_{D^*})^* = (e_{L^X}(A, D^*) \odot A)^* \\ &= e_{L^X}(A, D^*) \to A^*. \end{aligned}$$

Let $B(a) = 0.7, B(b) = 0.5$ be given.

(3) We can find the least element $\alpha = 0.9$ such that $B \leq \alpha \to A$. Hence

$$C_{\mathcal{M}}(B) = 0.9 \to A = (0.7, 0.9).$$

(4) We can find the greatest element $\alpha = 0.7$ such that $\alpha \odot A \leq B$. Hence

$$I_{\mathcal{K}}(B) = 0.7 \odot A = (0.3, 0.5).$$

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