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THE TIME SHIFTING THEOREM AND THE CONVOLUTION FOR ELZAKI TRANSFORM

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Abstract: Integral transform methods have been modified to solve the several dynamic equations with initial or boundary conditions in many ways. Laplace, Sumudu and Elzaki transforms are such typical things. Among these, Elzaki transform is an efficient and novel tool. In this article, we have proposed the solution of differential equation with variable coefficients using Elzaki transform, and proved the time shifting theorem and the convolution one for Elzaki transform.

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1. Introduction

The Laplace transform method is performing a worthy role for solving linear ordinary differential equations and corresponding initial value problems, having an idea to replace operations of calculus on functions with operations of algebra on transforms, but it is not appropriate for differential equations with variable coefficients.

In order to overcome this limitation, many researches[1, 8-12, 16, 20] for integral transform have been pursued such as Sumudu transform and Elzaki one, and they applied to the transform for solving differential equations with

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variable coefficients. Sumudu transform/Elzaki transform, a modified Laplace transform, was introduced by Watugala in 1993/Elzaki on 2011 to solve initial value problems in control engineering problems, and it is defined by

$$F(u) = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)dt/$$

$$T(u) = u\int_0^\infty e^{-t/u} f(t)dt,$$

respectively. This slight difference makes many change. Elzaki insists that the Elzaki transform can be easily applied to the initial value problems with less computational work, and can solve various examples which is not solved by the Laplace or Sumudu transform. By the reason, we would like to propose the solution of differential equation with variable coefficients using Elzaki transform, and investigate some examples and theorems(time shifting/convolution) for Elzaki transform.

We have cognizance that the Elzaki transform is well applied for initial value problems with variable coefficients and solving integral equations of convolution type, despite of this may have the maladapted disadvantage for finding a general solution.

In this article, we have showed Elzaki transform of Euler-Cauchy equation, the Volterra integral equation of the second kind, its convolution theorem, and time shifting one.

2. The Solution of Euler-Cauchy Equation using Elzaki Transform

The topic for differential equation with variable coefficients[3-7, 18-21] has aroused deep interest of many researchers. In relation with this, let us check the definition of Elzaki transform.

Definition 1. The Elzaki transform of the functions belonging to a class A, where $A = \{f(t)|\exists M, k_1, k_2 > 0 \ such\ that\ |f(t)| < Me^{|t|/k_j},\ if\ t \in (-1)^j \times [0,\infty)\}$ where $f(t)$ is denoted by $E[f(t)] = T(u)$ and defined as

$$T(u) = u^2\int_0^\infty f(ut)e^{-t}dt, \qquad k_1, k_2 > 0,$$

or equivalently,

$$T(u) = u\int_0^\infty f(t)e^{-t/u}dt, \quad u \in (k_1, k_2).$$

We can naturally obtain the following results from the definition and simple calculations.

$$
\begin{aligned}
&1)E[f'(t)] = T(u)/u - uf(0)\\
&2)E[f''(t)] = T(u)/u^2 - f(0) - uf'(0)\\
&3)E[tf'(t)] = u^2\frac{d}{du}[T(u)/u - uf(0)] - u[T(u)/u - uf(0)]\\
&4)E[t^2f'(t)] = u^4\frac{d^2}{du^2}[T(u)/u - uf(0)]\\
&5)E[tf''(t)] = u^2\frac{d}{du}[T(u)/u^2 - f(0) - uf'(0)]\\
&\quad -u[T(u)/u - f(0) - uf'(0)]\\
&6)E[t^2f''(t)] = u^4\frac{d^2}{du^2}[T(u)/u^2 - f(0) - uf'(0)].
\end{aligned}
$$

for $E(f(t)) = T(u)$[8-12].

On the other hand, Euler-Cauchy equations are ordinary differential equations of the form $t^2y'' + aty' + by = 0$ with given constants a and b and unknown $y(t)$[13, 15]. And the equation appears in a numbers of physics and engineering applications, such as when solving Laplace's equation in a polar coordinates, describing time-harmonic vibrations of a thin elastics rod, boundary value problem in spherical coordinates and so on[17]. Since the equation consist of variable coefficients, the Laplace transform method is not appropriate for this.

Theorem 1. *The solution of Euler-Cauchy equation $t^2y'' + aty' + by = 0$ can be expressed by*

$$y = \frac{(a-3)f(0) + tf'(0)}{b - a + 3}$$

using Elzaki transform.

Proof. Let us take Elzaki transform on both sides. Then

$$
\begin{aligned}
E(t^2y'') &= u^4\frac{d^2}{du^2}[T(u)/u^2 - f(0) - uf'(0)]\\
&= u^4\frac{d}{du}\,[\frac{T'(u)u - 2T(u)}{u^3} - f'(0)]\\
&= T''(u)u^2 - 4uT'(u) + 6T(u),
\end{aligned}
$$

$$E(aty') = au^2\frac{d}{du}[T(u)/u - uf(0)] - au[T(u)/u - uf(0)]$$

$$= au^2[\frac{T'(u)u - T(u)}{u^2} - f(0)]$$
$$= auT'(u) - 2aT(u)$$

and $E[by] = bT(u)$. Arranging the equality, we get

$$u^2T''(u) + (a-4)uT'(u) + (b-2a+6)T(u) = 0$$

where $E(f(t)) = T(u)$ is Elzaki transform of $f(t)$. Thus, we have

$$u^2[T(u)/u^2 - f(0) - uf'(0)] + (a-4)u[T(u)/u - uf(0)]$$
$$+(b-2a+6)T(u) = 0.$$

Collecting the $T(u)$-terms, we get

$$(b-a+3)T(u) - (a-3)f(0)u^2 - f'(0)u^3 = 0.$$

Since $E(1) = u^2$ and $E(t) = u^3$, we obtain

$$y = T^{-1}(u) = \frac{(a-3)f(0) + tf'(0)}{b-a+3}.$$

□

In [17], we proposed that a solution of Euler-Cauchy equation $t^2y'' + aty' + by = 0$ using Laplace transform can be expressed by

$$y = my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$
$$+ \{ay(0)(m/2-1) + my'(0)\}\ (\ 1/\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ )e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$

where $d/ds = m$. The result of theorem 1 is simpler than that of Laplace transform.

3. The Time Shifting Theorem and the Convolution for Elzaki Transform

Let us begin the lemma 1.

Lemma 1. Let $T(u)$ be Elzaki transform of the function $f(t)$ in $A = \{f(t)|\exists M, k_1, k_2 > 0\ such\ that\ |f(t)| < Me^{|t|/k_j},\ if\ t \in (-1)^j \times [0,\infty)\}$ with

Laplace transform $F(s)$. Then Elzaki transform $T(u)$ of $f(t)$ is given by $T(u) = uF(\frac{1}{u})$ [12].

Next, we consider convolution theorem for Elzaki transform. Elzaki[9] showed convolution theorem for Elzaki transform using lemma 1, and Ali[2] showed the theorem using infinite convergent series, but we would like to propose some more general manners as follows;

Theorem 2. (Convolution theorem for Elzaki transform)

$$E(f * g) = \frac{1}{u} E(f)E(g)$$

for $E(f)$ is the Elzaki transform of f.

Proof.

$$E(f)E(g) = u^2 \int_0^\infty f(\tau)e^{-\tau/u}d\tau \cdot \int_0^\infty g(v)e^{-v/u}dv. \qquad (*)$$

As let us put $t = v + \tau$, $v = t - \tau$ and so, we get

$$E(g) = \int_\tau^\infty g(t-\tau)e^{-(t-\tau))/u}dt$$

$$= e^{\tau/u} \int_\tau^\infty e^{-t/u} g(t-\tau)dt.$$

Thus

$$(*) = u^2 \int_0^\infty f(\tau)e^{-\tau/u}e^{\tau/u} \int_\tau^\infty e^{-t/u}g(t-\tau)dtd\tau$$

$$= u^2 \int_0^\infty f(\tau) \int_\tau^\infty e^{-\tau/u}g(t-\tau)dtd\tau$$

$$= u^2 \int_0^\infty e^{-t/u} \int_0^t f(\tau)g(t-\tau)d\tau dt$$

$$= u^2 \int_0^\infty e^{-t/u} \cdot (f * g)(t)dt$$

$$= uE(f * g).$$

□

Example 1. Solve the Volterra integral equation of the second kind

$$y(t) - \int_0^t y(\tau)sin(t-\tau)d\tau = t.$$

In [14/ p253], we can find the solution by the Laplace transform,

$$y(t) = t + \frac{t^3}{6}.$$

Next, let us approach by the Elzaki transform. The equation can be written by

$$y - y * sint = t.$$

Writing $Y = E(y)$ and applying the convolution theorem, we obtain

$$Y(u) - \frac{1}{u}Y(u)\frac{u^3}{1+u^2} = u^3$$

where

$$E(sin\ at) = \frac{au^3}{1+a^2u^2} \quad and \quad E(t) = u^3 \quad [12].$$

Thus, the solution is

$$Y(u)(1 - \frac{u^2}{1+u^2}) = u^3$$

and arranging the equation,

$$Y(u) \ = u^3(1+u^2) = u^3 + u^5.$$

As we scan a table of Elzaki transforms[12], we see that the answer

$$y(t) = t + \frac{t^3}{6}.$$

In the above example, we have obtained the same result in both transforms. However, there are several examples can solve only by Elzaki transform.

Example 2. The equation $y'' + ty' - y = 0$, $y(0) = 0$, $y'(0) = 1$ for $t > 0$ cannot solve by Laplace transform, but can solve by Elzaki one.

The equation is represented by

$$\frac{ds}{s} = \frac{dY}{s^2Y - 2Y - 1}$$

using Laplace transform for $Y = \pounds(y)$ but we cannot find the solution. However, if we use Elzaki transform,

$$\frac{T(u)}{u^2} - y(0) - uy'(0) + u^2\frac{d}{du}[\frac{T(u)}{u} - uy(0)]$$

$$-u[\frac{T(u)}{u} - uy(0)] - T(u) = 0$$

for $T(u) = E(y)$. Using the initial conditions, we have

$$\frac{T(u)}{u^2} - u + u^2\frac{d}{du}(\frac{T(u)}{u}) - 2T(u) = 0.$$

By the simple calculation, we get

$$T'(u) + (\frac{1}{u^3} - \frac{3}{u})T(u) = 1.$$

Implies, $T(u) = u^3 + cu^3e^{1/2u^{\mathbf{2}}}$ and $c = 0$. Hence, $T(u) = u^3$ and by the inverse Elzaki transform, we get the solution $u = t$.

Example 3. There exists several examples can solve by Elzaki transform but cannot solve by Laplace or Sumudu transform[12].

$$1)\ \ t^2y'' + 4ty' + 2y = 12t^2,\ y(0) = y'(0) = 0$$

$$2)\ \ t^2y''' + 6ty'' + 6y' = 60t^2,\ y(0) = y'(0) = y''(0) = 0$$

The above example do not give a guarantee that Elzaki transform better than the other ones. However, Elzaki transform is successfully applied to integral equation and initial value problems at $t = 0$, despite of this have the difficulty for finding a general solution.

Note that the Elzaki transform of unit step function $E\{u(t-a)\}$ is $u^2e^{-a/u}$ for $u > 0$. Using this, we would like to propose time shifting theorem for Elzaki transform.

Theorem 3. (Time shifting theorem for Elzaki transform) *If* $E\{f(t)\} = T(u)$, *then*

$$E\{f(t-a)u*(t-a)\} = e^{-a/u}T(u)$$

for u(t) is the unit step function.*

Proof. The shifted function $E\{f(t-a)u*(t-a)\}$ is defined by

$$u\int_a^\infty e^{-t/u}f(t-a)dt$$

and putting $\tau = t-a$, we obtain

$$E\{f(t-a)u(t-a)\} = u\int_0^\infty e^{-\frac{\tau+a}{u}}f(\tau)d\tau$$

$$= ue^{-a/u}\int_0^\infty e^{-\tau/u}f(\tau)d\tau$$

$$= e^{-a/u}u\int_0^\infty e^{-\tau/u}f(\tau)d\tau = e^{-a/u}T(u).$$

□

Example 4.

$$E\{5\ sin(t-2)u*(t-2)\} = 5e^{-2/u}\frac{u^3}{1+u^2}$$

for u*(t) is the unit step function.

It is clear from

$$E(5\ sint) = \frac{5u^2}{1+u^2}.$$

Let us check the above example by a direct calculation. Putting $I = E\{5\ sin(t-2)u*(t-2)\}$,

$$I = 5u\int_2^\infty e^{-t/u}sin(t-2)dt$$

$$= 5u\{[-ue^{-t/u}sin(t-2)]_2^\infty + u\int_2^\infty e^{-t/u}cos(t-2)dt\}.$$

Since the first term is 0, let us integrate by part to the second term. Then we get

$$I = 5u^2\{[-ue^{-t/u}cos(t-2)]_2^\infty - u\int_2^\infty e^{-t/u}sin(t-2)dt\}$$

$$= 5u^2(ue^{-2/u} - 1/5I).$$

Hence, we can easily get the result

$$I = 5e^{-2/u}\frac{u^3}{1+u^2}.$$

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