International Journal of Pure and Applied Mathematics
Volume 87 No. 3 2013, 487-495
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v87i3.12
ijpam.eu

SEVERAL REPRESENTATIONS OF THE EULER-CAUCHY EQUATION WITH RESPECT TO INTEGRAL TRANSFORMS

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Abstract: The subjects on the solution of differential equations with variable coefficients have aroused the interest of many researchers. Existing many integral transforms are playing a role to get the solution of it. In this sense, we have researched about several forms of Euler-Cauchy equation, using Laplace transform. Related to the topic, the proposed idea can be also applied to other transforms (Sumudu/Elzaki, etc.).

AMS Subject Classification: 44A10, 34A12, 65R10
Key Words: variable coefficient, Euler-Cauchy equation, Laplace/Sumudu/Elzaki transform

1. Introduction

Euler-Cauchy equation appears in a number of physics and engineering applications, when solving Laplace's equation in a polar coordinates, describing time-harmonic vibrations of a thin elastics rod and boundary value problem in spherical coordinates, and so on [14]. The equation has a basic form of $t^2y'' + aty' + by = r(t)$, and this is considered as an easy type among differential equations with variable coefficients. Hence we would like to consider the Euler-

Received: August 9, 2013

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Cauchy equation as a shortcut to study properties of differential equation with variable coefficients. In [14], we have proposed that a solution of homogeneous Euler-Cauchy equation with an initial condition can be expressed by

$$y = my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m}-\frac{a^2}{4}}\,t$$
$$+\{ay(0)(m/2-1)+my'(0)\}\ (\ 1/\sqrt{\frac{b}{m}-\frac{a^2}{4}}\ )e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m}-\frac{a^2}{4}}\,t,$$

where $d/ds = m$.

We would like to take a look into Preceding researches of this area. Watugula [20] introduced the Sumudu transform in order to apply it to find the solution of ordinary differential equations in control engineering problems on 1993. Agwa dealt with Sumudu transform on time scales in [1], and there are many researches related to the transform[2-4, 11-13]. Recently(2011) Elzaki proposed Elzaki transform[6-10] defined by

$$T(u) = u\int_0^\infty e^{-t/u}f(t)dt$$

for $E[f(t)] = T(u)$, and he says that it efficiently be used to solve problems without resorting to anew frequency domain because it preserve scales and units properties [7]. We have checked it to find the solution of differential equations with variable coefficients[15], and have found it has a strong point managing differential equations with variable coefficients compared with existing other integral transforms. On the other hand, efforts to find solutions of differential equations with variable coefficients using integral transforms have been pursued[4,14-15,17-19].

In this article, we have checked several representations of Euler-Cauchy equation with an initial condition and the validity of the solution of it using the differentiation of transforms.

2. Several Representations of the Euler-Cauchy Equation

First, let us check representations of Euler-Cauchy equation with an initial condition. Without repetitive description, we will use $\pounds$ as a symbol of Laplace transform.

Theorem 1. *The Laplace transform of Euler-Cauchy equation* $t^2y'' + aty' + by = r(t)$*[16] can be expressed by*

$$Y = \frac{1}{b}\ [R - ne^{-ns}(ny'(n) + ay(n)) \\ - \int_0^n e^{-st}\{(st-2)ty'(t) + a(st-1)y(t)\}\ dt],$$

for $t < n$, $Y = \pounds(y)$ *and* $R = \pounds(r)$.

Proof. By mathematical induction,

$$\pounds(y') = e^{-ns}y(n) - y(0) + s\int_0^n e^{-st}y(t)\ dt$$

holds for $t < n$ and for n is a natural number[17]. Substituting y' to y, we have

$$\pounds(y'') = e^{-ns}y'(n) - y'(0) + s\int_0^n e^{-st}y'(t)\ dt$$

and so, following equalities are hold

$$\begin{aligned}\pounds(ty') =& - d/ds[e^{-ns}y(n) - y(0) + s\int_0^n e^{-st}y(t)\ dt] \\ =& - [-ne^{-ns}y(n) + \int_0^n e^{-st}y(t)\ dt - st\int_0^n e^{-st}ty(t)\ dt] \\ =& ne^{-ns}y(n) + \int_0^n e^{-st}(st-1)y(t)\ dt,\end{aligned}$$

$$\begin{aligned}\pounds(ty'') =& - d/ds[\pounds(y'')] \\ =& - d/ds[e^{-ns}y'(n) - y'(0) + s\int_0^n e^{-st}y'(t)\ dt] \\ =& ne^{-ns}y'(n) + \int_0^n e^{-st}(st-1)y'(t)\ dt,\end{aligned}$$

and

$$\begin{aligned}\pounds(t^2y'') =& - d/ds\{\pounds(ty'')\} \\ =& - d/ds\ [ne^{-ns}y'(n) + \int_0^n e^{-st}(st-1)y'(t)\ dt] \\ =& -\ [-n^2e^{-ns}y'(n) - t\int_0^n e^{-st}(st-1)y'(t)\ dt \\ & + \int_0^n e^{-st}ty'(t)\ dt]\end{aligned}$$

$$=n^2e^{-ns}y'(n)+\int_0^n e^{-st}(st-2)ty'(t)\ dt.$$

Thus, the given Euler-Cauchy equation can be expressed by

$$n^2e^{-ns}y'(n)+\int_0^n e^{-st}(st-2)ty'(t)\ dt$$
$$+a[\ ne^{-ns}y(n)+\int_0^n e^{-st}(st-1)y(t)\ dt]+bY=\ R,$$

where $R=\pounds(r)$. Organizing the equality, we have

$$Y=\frac{1}{b}\ [R-ne^{-ns}(ny'(n)+ay(n))$$
$$-\int_0^n e^{-st}\{(st-2)ty'(t)+a(st-1)y(t)\}\ dt],$$

for $t<n$, $Y=\pounds(y)$ and $R=\pounds(r)$. □

Next, we would like to propose the expression using the integration for the Euler-Cauchy equation.

Theorem 2. *In nonhomogeneous Euler-Cauchy equation* $t^2y''+aty'+by=r(t)$*:*

(1) its initial condition can be expressed by $y(0)=2sY$ *and* $y'(0)=-2s^2Y$ *for* $a=2$ *and for* $Y=\pounds(y)=F(s)$.

(2)
$$Y=\frac{3}{2}(a-4)\ln s-\frac{3}{2}(a-b-2)\pounds\{y(t)/t\}\frac{1}{s^2}$$
for $\ln s$ *is the natural logarithm.*

(3) a solution can be expressed by the form

$$y=my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m}-\frac{a^2}{4}}\ t$$
$$+\ \{ay(0)(m/2-1)+my'(0)\}\ (\ 1/\sqrt{\frac{b}{m}-\frac{a^2}{4}}\ )e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m}-\frac{a^2}{4}}\ t$$
$$+\frac{1}{m}\pounds^{-1}[\frac{R}{s^2-sa+b/m}]$$

for $R=\pounds(r)$ *and for* $d/ds=m$.

Proof. (1) The Laplace transform of Euler-Cauchy equation $t^2y'' + aty' + by = r(t)$ has been expressed by

$$s^2\frac{d^2Y}{ds^2} + (4s - as)\frac{dY}{ds} + (b - a + 2)Y = R, \qquad (*)$$

or equivalently

$$\frac{d^2}{ds^2}[s^2Y - sy(0) - y'(0)] - a\frac{d}{ds}[sY - y(0)] + bY = R \qquad (**)$$

for $R = \pounds(r)$[12]. Since the equation (*) is identical with (**), we get

$$s^2Y = s^2Y - sy(0) - y'(0)$$

and

$$(4s - as)Y = -a(sY - y(0))$$

for $a = 2$. By the simple calculation, we have $y(0) = 2sY$ and $y'(0) = -2s^2Y$ for $a = 2$.

(2) Let us integrate the above equality (*) from s to ∞. Then we have

$$\frac{1}{3}s^3\frac{dY}{ds} + (4 - a)(1/2)s^2 + (b - a + 2)\int_s^\infty Y ds = Rs.$$

Arranging the equality and using $\int_s^\infty Y ds = \pounds\{y(t)/t\}$, we have

$$\frac{dY}{ds} = 3/s^3[(a - 4)(1/2)s^2 + (a - b - 2)\pounds\{y(t)/t\} + Rs].$$

Thus

$$dY = [\frac{3(a-4)}{2s} + \frac{3(a-b-2)\pounds\{y(t)/t\}}{s^3} + \frac{3R}{s^2}]\ ds.$$

Integrating both sides, we have

$$Y = \frac{3}{2}(a - 4)lns - \frac{3}{2}(a - b - 2)\pounds\{y(t)/t\}\frac{1}{s^2} - \frac{3R}{s}$$

for lns is the natural logarithm.

(3) This is an immediate consequence of [14](see Theorem 2). □

Definition 3. The Elzaki transform of the functions belonging to a class A, where $A = \{f(t)|\exists M, k_1, k_2 > 0 \; such \; that \; |f(t)| < Me^{|t|/k_j}, \; if \; t \in (-1)^j \times [0, \infty)\}$ where $f(t)$ is denoted by $E[f(t)] = T(u)$ and defined as

$$T(u) = u^2 \int_0^\infty f(ut)e^{-t}dt, \qquad k_1, k_2 > 0,$$

or equivalently,

$$T(u) = u \int_0^\infty f(t)e^{-t/u}dt, \quad u \in (k_1, k_2)[7].$$

Theorem 4. *The solution of nonhomogeneous Euler-Cauchy equation* $t^2y'' + aty' + by = r(t)$ *can be expressed by*

$$y = \frac{(a-3)f(0) + tf'(0) + r(t)}{b-a+3},$$

and it has a validity in terms of differentiation of transforms.

Proof. In [15], we have already checked the solution, using Elzaki transform as following;

$$(b-a+3)T(u) - (a-3)f(0)u^2 - f'(0)u^3 = R$$

for $R = T(r)$ and $T(u)$ is Elzaki transform of $y(t)$. Since $E(1) = u^2$ and $E(t) = u^3$, we have

$$y = \frac{(a-3)f(0) + tf'(0) + r(t)}{b-a+3}.$$

Next, we would like to check the validity of the solution using $\pounds^{-1}\{F'(s)\} = -tf(t)$. Since $F(s) = s\,T(1/s)$, it is clear that its Laplace transform is the form of

$$F(s) = \frac{(a-3)f(0)(1/s) + f'(0)(1/s^2) + R}{b-a+3}$$

for $F(s)/T(u)$ is Laplace/Elzaki transform of $y(t)$, respectively. Differentiating $F(s)$ under the integral sign with respect to s, we have

$$F'(s) = \frac{-(a-3)f(0)(1/s^2) - 2f'(0)(1/s^3) + dR/ds}{b-a+3}.$$

Taking the inverse transform, we obtain

$$\pounds^{-1}\{F'(s)\} = \frac{-(a-3)f(0)t - f'(0)t^2 - tr}{b-a+3} = -ty(t).$$

Hence the solution

$$y = \frac{(a-3)f(0) + tf'(0) + r(t)}{b-a+3}$$

has a validity in terms of differentiation of transforms. □

It is very difficult or impossible to solve Euler-Cauchy equation $t^2y'' + aty' + by = r(t)$ by Laplace transform directly. Hence, we have obtained the solution by Elzaki transform. It is easy to change from Elzaki transform to Laplace's thanks to the interaction formula $F(s) = s\ T(1/s)$ for $F(s)/T(u)$ is Laplace/Elzaki transform of $y(t)$, respectively.

For example, let us consider the initial value problem $y'' + ty' - y = 0$, $y(0) = 0$, $y'(0) = 1$. Since $a = 1$ and $b = -1$, by theorem 4, we have

$$y = -2 \cdot 0 + t/{-1} - 1 + 3.$$

Thus $y = t$.

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