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SIMULATION SUPPLEMENT TO GOODNESS-OF-FIT TESTS DERIVED FROM CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS VIA RECORD VALUES

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Abstract: We give simulations of powers of goodness-of-fit tests for normality derived from characterizations of continuous distributions via record values and presented in Morris and Szynal [3].

AMS Subject Classification: 62G10, 62G30
Key Words: order statistics, record values, U-statistics, exponential and normal distributions, characterizations, goodness-of-fit tests, powers

1. Introduction

Morris and Szynal [3] constructed goodness-of-fit tests for continuous distributions using characterizations of continuous distributions via expected values of two functions of record values and U-statistics. In particular, tests for exponentiality, Rayleigh distribution and normality were presented. Simulations of powers of tests for exponential and Rayleigh distribution were also obtained to compare them with other tests. The aim of this contribution is to give simulations of powers of tests for normality to compare them with some recommended tests.

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Tests and alternatives are taken from Pearson et al [4] as it was done in Morris and Szynal [2] where tests for normality are based on characterizations involving moments of order statistics.

Additionaly we choose for comparisons tests and alternatives studied in Kallenberg and Ledwina [1].

Simulations show that our tests for normality perform very well and can be applied instead of classical recommended tests.

2. Tests for Normality

Tests of $H: \ X \sim N(\mu, \sigma)$ we will discuss are presented in Morris and Szynal [3]. Here by f and F we denote corresponding density and the cumulative distribution function, $\Lambda = \{(\mu, \sigma^2): \ \mu \in \mathbf{R}, \sigma^2 > 0\}$, i.e.

$$\begin{aligned} f(x; \mu, \sigma^2) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \\ F(x) &= \int_{-\infty}^{x} f(t)dt, \quad -\infty < x < \infty. \end{aligned}$$

For $X \sim N(0,1)$ we write

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Moreover, we use

$$\hat{\mu}_n = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2, \quad S_n = \sqrt{\hat{\sigma}_n^2}.$$

The following quantities are appearing in the formulae of test-statistics:

$$\begin{aligned} a_n^{(r,k)} &= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2\frac{\Gamma(2r+2)}{k^r(k-j)^r} B_{\frac{k-j}{2k-j}}(r+1, r+1)\right. \\ &+ \left. j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+1) - \Gamma^2(r+1)}{k^{2r}}\right], \\ b_n^{(r,k)} &= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2(2k-j)\frac{\Gamma(2r+2) + \Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}}\right. \\ &\qquad \left. B_{\frac{k-j}{2k-j}}(r+2, r+2) + j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right] \\
c_n^{(r,k)} &= \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\left[2\frac{\Gamma(2r+4)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2)\right. \\
&+ \left. j\frac{\Gamma(2r+3)}{(2k-j)^{2r+3}}-\frac{\Gamma^2(r+2)}{k^{2r+2}}\right]+\frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+3)-\Gamma^2(r+2)}{k^{2r+2}}\right],
\end{aligned}
$$

where

$$B_x(\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0<x<1; \quad \alpha,\beta>0,$$

denotes the incomplete beta function.

$$
\begin{aligned}
E_1^{(r,k)} &= \mathrm{E}\left[\phi(Z)(1-\Phi(Z))^{k-2}\log^{r-1}\frac{1}{1-\Phi(Z)}\right], \quad Z\sim N(0,1), \\
E_2^{(r,k)} &= \mathrm{E}\left[Z\phi(Z)(1-\Phi(Z))^{k-2}\log^{r-1}\frac{1}{1-\Phi(Z)}\right].
\end{aligned}
$$

$$
\begin{aligned}
s_n^{(r,k)} &= \frac{k^2r^2}{n}\left[(E_1^{(r,k)})^2+\frac{1}{2}(E_2^{(r,k)})^2\right] \\
t_n^{(r,k)} &= \frac{k^2r(r+1)}{n}\left[E_1^{(r,k)}E_1^{(r+1,k)}+\frac{1}{2}E_2^{(r,k)}E_2^{(r+1,k)}\right] \\
u_n^{(r,k)} &= \frac{k^2(r+1)^2}{n}\left[(E_1^{(r+1,k)})^2+\frac{1}{2}(E_2^{(r+1,k)})^2\right] \\
a_{n1}^{(r,k)} &= a_n^{(r,k)}-s_n^{(r,k)} \\
b_{n1}^{(r,k)} &= b_n^{(r,k)}-t_n^{(r,k)} \\
c_{n1}^{(r,k)} &= c_n^{(r,k)}-u_n^{(r,k)}
\end{aligned}
$$

$$\Delta_{n1}^{(r,k)}=\det\begin{bmatrix} a_{n1}^{(r,k)} & b_{n1}^{(r,k)} \\ b_{n1}^{(r,k)} & c_{n1}^{(r,k)}\end{bmatrix}.$$

The proposed in Morris and Szynal [3] test-statistics for normality are as follows

$$
\begin{aligned}
\hat{T}_n^{(r,k)} &= \frac{1}{\Delta_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}\right.\right. \\
&- \left.\frac{\Gamma(r+1)}{k^r}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&- 2b_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}-\frac{\Gamma(r+1)}{k^r}\right)\\
&\times\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)\\
&+ a_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)\Bigg]\\
&= \hat{T}_{n;c_1}^{(r,k)}+\hat{T}_{n;c_2}^{(r,k)}=\hat{T}_{n;c_3}^{(r,k)}+\hat{T}_{n;c_4}^{(r,k)},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{T}_{n;c_1} &= \frac{1}{a_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}-\frac{\Gamma(r+1)}{k^r}\right]^2\\
\hat{T}_{n;c_2} &= \frac{1}{\Delta_{n1}^{(r,k)}a_{n1}^{(r,k)}}\left[a_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}\right.\\
&- b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}\\
&- \left.\left(a_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}-b_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^r}\right)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{T}_{n;c_3} &= \frac{1}{c_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right]^2\\
\hat{T}_{n;c_4} &= \frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}\right.\\
&- b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)}\\
&- \left.\left(c_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^{r}}-b_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}\right)\right]^2.
\end{aligned}
$$

3. Simulation Results

For an empirical comparison of the performances of our tests we use some alternatives and tests choosing from Pearson et al [4] with numbering using there (cf. Table 1) and studied in Morris and Szynal [2]. Here we do also comparison of the performances of our tests with the data driven smooth tests of Kallenberg and Ledwina [1] (cf. Table 2).

We discuss the following omnibus tests:

K^2 :	The Brown & Shenton test,
R :	The rectangle test,
S-W:	The Shapiro-Wilk test (W in [4]),
Y :	The D'Agostino test,
W :	The Shapiro-Wilk test (W in [1]),
W_S :	The data driven smooth test statistic,
W_{S1} :	The data driven smooth modified test statistic,
W_{S2} :	The data driven smooth test statistic without "adjustment".

When $n = 20$ and $n = 50$ the powers of the test $\hat{T}_n^{(r,k)}$ and their components $\hat{T}_{n;c_1}^{(r,k)}$, $\hat{T}_{n;c_2}^{(r,k)}$, $\hat{T}_{n;c_3}^{(r,k)}$, $\hat{T}_{n;c_4}^{(r,k)}$ were investigated for $r = -0.499$, -0.45, -0.4, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 and $k = 1, 2, 3, 4, 5$. Critical values were simulated using 100 000 samples and the associated power were obtained using 100 000 samples, but only some results are presented here.

The alternatives chosen in Morris and Szynal [2] we write Alt.M-S while the alternatives used in Kallenberg and Ledwina [1] we denote Alt.K-L.

For samples of size $n = 20$ we include simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 34.0 (Alt.M-S); ≥ 37 (Alt.K-L) and under skew alternatives with Av. powers ≥ 50.0 (Alt.M-S); ≥ 44 (Alt.K-L) (Tables 3-6).

For samples of size $n = 50$ we include simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 62.0 (Alt.M-S); ≥ 69 (Alt.K-L) and under skew alternatives with Av. powers ≥ 83.0 (Alt.M-S); ≥ 73 (Alt.K-L) (Tables 7-10).

$n=20$					
symmetric					
	Alt.M-S	K^2	R	S-W	Y
1	$SB(0;0.5)$	38	40	48	8
4	$SB(0;0.707)$	14	16	20	8
7	$Beta(2;2)$	14	4	2	7
15	$Logistic(1)$	16	12	12	11
18	Laplace	31	26	26	28
21	$SC(0.1;3)$	36	34	26	28
24	$SC(0.1;5)$	61	60	50	48
27	$SC(0.05;7)$	49	48	42	41
30	$t(4)$	25	24	25	26
32	$t(1)$	79	80	91	92
	Av.	36.3	34.4	34.2	29.7
skew					
	Alt.M-S	K^2	R	S-W	Y
36	$SB(0.533;0.5)$	29	29	75	11
38	$Beta(2;1)$	12	11	29	6
40	$SB(1;1)$	18	18	36	11
42	$LC(0.2;5)$	43	43	86	48
44	$Weibull(2)$	14	14	19	7
50	χ^2_4	40	39	46	24
52	$LC(0.05;5)$	53	51	59	52
54	$LC(0.05;7)$	61	60	63	62
56	$Weibull((0.5)$	96	96	100	96
58	$LN(0;1)$	82	81	83	77
	Av.	44.8	44.2	59.6	39.4

$n=50$					
symmetric					
	Alt.M-S	K^2	R	S-W	Y
1	$SB(0;0.5)$	96	96	100	43
4	$SB(0;0.707)$	52	58	70	61
7	$Beta(2;2)$	18	17	23	32
15	$Logistic(1)$	23	22	8	22
18	Laplace	54	54	39	56
21	$SC(0.1;3)$	61	58	48	58
24	$SC(0.1;5)$	88	88	80	86
27	$SC(0.05;7)$	80	80	73	74
30	$t(4)$	46	43	36	49
32	$t(1)$	98	98	99	100
	Av.	61.6	61.4	57.6	58.1
skew					
	Alt.M-S	K^2	R	S-W	Y
36	$SB(0.533;0.5)$	82	82	100	18
38	$Beta(2;1)$	32	31	91	12
40	$SB(1;1)$	35	35	86	10
42	$LC(0.2;5)$	92	94	100	84
44	$Weibull(2)$	33	32	38	12
50	χ^2_4	80	82	96	62
52	$LC(0.05;5)$	90	89	82	82
54	$LC(0.05;7)$	93	93	90	90
56	$Weibull((0.5)$	100	100	100	100
58	$LN(0;1)$	99	100	100	99
	Av.	73.6	73.8	88.3	56.9

Table 1: (Source: Pearson et al [4]) Empirical comparision of the performances of the tests K^2, R, S-W and Y based on 200 replications. The entries are simulated powers of 5% tests.

$n = 20$								
symmetric								
	Alt.K-L	W	W_S	W_{S1}	W_{S2}	K^2	R	Y
1	$SB(0;0.5)$	44	36	34	26	38	40	8
2	$TU(1.5)$	26	20	19	14	19	20	12
5	$TU(0.7)$	12	11	9	8	10	12	12
15	$Logistic(1)$	12	10	13	11	16	12	11
17	$TU(10)$	82	82	87	85	46	45	84
20	$SC(0.05;3)$	19	17	19	16	24	22	20
22	$SC(0.2;5)$	71	65	74	65	69	68	70
25	$SC(0.05;5)$	36	33	37	32	39	38	34
27	$SC(0.05;7)$	45	42	46	42	49	48	41
28	$SU(0,1)$	43	36	47	38	44	42	42
	Av.	39.0	35.2	38.5	33.7	35.4	34.7	33.4
skew								
	Alt.K-L	W	W_S	W_{S1}	W_{S2}	K^2	R	Y
36	$SB(0.533;0.5)$	73	63	47	55	29	29	11
40	$SB(1;1)$	31	29	17	29	18	18	11
41	$LC(0.2;3)$	31	27	19	28	16	16	9
44	$Weibull(2)$	15	15	10	16	14	14	7
45	$LC(0.1;3)$	25	24	21	26	30	28	12
46	χ^2_{10}	25	23	18	26	27	25	16
47	$LC(0.05;3)$	18	17	18	17	26	25	12
48	$LC(0.1;5)$	76	72	72	73	64	63	59
49	$SU(-1;2)$	22	19	20	21	20	20	18
50	χ^2_4	53	51	38	52	40	39	24
52	$LC(0.05;5)$	55	48	54	49	53	51	52
54	$LC(0.05;7)$	65	64	65	63	61	60	62
57	$SU(1;1)$	73	73	68	73	68	68	60
58	$LN(0;1)$	94	91	85	92	82	81	77
	Av.	46.9	44.0	39.4	44.3	39.1	38.4	30.7

$n = 50$								
symmetric								
	Alt.K-L	W	W_S	W_{S1}	W_{S2}	K^2	R	Y
1	$SB(0;0.5)$	99	67	93	55	96	96	43
2	$TU(1.5)$	92	34	74	26	78	80	58
5	$TU(0.7)$	62	13	45	9	47	48	61
15	$Logistic(1)$	13	13	21	12	23	22	22
17	$TU(10)$	99	99	100	99	58	66	100
20	$SC(0.05;3)$	31	25	38	24	44	40	38
22	$SC(0.2;5)$	95	92	98	92	96	97	98
25	$SC(0.05;5)$	62	55	66	55	71	70	64
27	$SC(0.05;7)$	74	70	77	70	80	80	74
28	$SU(0,1)$	68	61	81	61	75	74	83
	Av.	69.5	52.9	69.3	50.3	66.8	67.3	64.1
skew								
	Alt.K-L	W	W_S	W_{S1}	W_{S2}	K^2	R	Y
36	$SB(0.533;0.5)$	100	96	95	92	82	82	18
40	$SB(1;1)$	81	72	57	71	35	35	10
41	$LC(0.2;3)$	60	68	52	69	29	31	18
44	$Weibull(2)$	41	41	29	41	33	32	12
45	$LC(0.1;3)$	50	58	51	58	50	50	38
46	χ^2_{10}	57	61	48	62	46	46	20
47	$LC(0.05;3)$	32	33	37	34	43	42	30
48	$LC(0.1;5)$	98	97	98	97	97	98	96
49	$SU(-1;2)$	37	43	40	42	46	44	27
50	χ^2_4	95	94	86	93	80	82	62
52	$LC(0.05;5)$	85	79	87	78	90	89	82
54	$LC(0.05;7)$	92	91	92	90	93	93	90
57	$SU(1;1)$	96	98	97	98	96	98	96
58	$LN(0;1)$	100	100	100	100	99	100	99
	Av.	73.1	73.6	69.2	73.2	65.6	65.9	49.9

Table 2: (Source: Kallenberg and Ledwina [1] and Pearson et al [4]) Empirical comparision of the performances of the tests W, W_S, W_{S1} and W_{S2} based on 10 000 replications and K^2, R and Y based on 200 replications. The entries are simulated powers of 5% tests.

$n=20$									
symmetric									
k	1							2	
Tests	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$
Alt.M-S/r	0.3	0.3	0.7	0.7	0.9	1.0	1.1	0.5	0.5
1	38	44	37	38	43	43	38	53	33
4	17	19	17	17	18	19	17	28	11
7	7	7	7	6	7	7	7	12	4
15	13	13	13	13	13	13	13	10	14
18	27	29	28	27	28	28	28	23	31
21	30	31	31	30	31	32	31	25	32
24	56	57	56	55	56	57	56	51	58
27	46	46	46	46	46	46	46	43	47
30	26	27	26	26	26	27	26	21	28
32	85	86	86	85	85	86	85	84	88
Av.	34.5	36.0	34.7	34.2	35.4	35.8	34.8	35.1	34.6

k	2			3				
Tests	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$
Alt.M-S/r	0.7	0.7	0.9	0.9	1.0	1.1	1.3	1.3
1	41	32	38	46	50	51	44	50
4	16	10	15	22	23	24	21	23
7	6	3	5	9	10	10	9	9
15	13	14	14	11	10	11	11	10
18	32	32	32	26	27	27	28	25
21	30	33	30	27	26	26	25	24
24	55	58	54	53	53	52	51	51
27	45	47	45	44	43	43	42	42
30	27	29	27	23	23	23	23	21
32	87	89	86	87	87	87	87	87
Av.	35.2	34.6	34.6	34.7	35.2	35.2	34.0	34.2

Table 3: Powers of 5% tests under symmetric alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 34.

$n=20$									
skew									
k	1	2					3		
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt.M-S/r	0.5	1.0	1.1	1.3	1.5	1.7	0.1	0.3	0.3
36	58	56	54	49	43	37	51	56	45
38	14	12	12	13	15	16	24	17	33
40	30	30	32	32	31	29	31	40	29
42	86	79	83	85	85	83	78	81	77
44	16	17	18	19	20	19	16	22	16
50	53	55	56	56	56	54	55	63	54
52	58	54	56	58	59	60	51	44	52
54	66	61	64	66	66	66	65	59	65
56	100	95	97	99	99	99	100	98	100
58	92	92	92	92	91	90	93	95	93
Av.	57.2	55.0	56.6	56.9	56.5	55.2	56.3	57.4	56.3
k	3	4				5			
Tests	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$
Alt.M-S/r	0.5	0.3	0.5	0.5	0.7	0.3	0.5	0.7	0.9
36	41	57	64	52	51	62	60	68	58
38	27	28	17	32	24	24	27	16	22
40	30	31	41	29	32	31	31	40	33
42	78	78	82	77	81	75	76	80	80
44	18	15	21	14	17	14	15	19	17
50	55	55	62	52	56	54	55	61	56
52	53	47	42	47	48	44	44	42	45
54	65	64	59	64	63	63	62	59	62
56	99	100	99	100	100	100	100	99	100
58	92	94	95	93	94	94	94	96	94
Av.	55.9	56.7	58.2	55.8	56.6	56.1	56.4	58.0	56.7

Table 4: Powers of 5% tests under skew alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 55.

$n=20$								
symmetric								
k	1		2			3		
Tests	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$
Alt.K-L/r	0.3	1.0	0.5	0.7	0.9	0.9	0.9	1.0
1	43	42	34	41	38	49	46	51
2	28	28	19	25	23	36	31	37
5	15	15	9	13	11	23	18	23
15	13	13	14	14	13	9	10	9
17	58	56	71	68	68	69	76	71
20	21	21	21	20	20	15	18	15
22	71	70	74	71	69	65	71	63
25	38	38	38	36	36	32	34	31
27	46	46	47	45	45	41	44	41
28	45	45	48	47	47	37	43	37
Av.	37.9	37.6	37.5	38.0	37.0	37.5	39.0	37.8
k	3						4	
Tests	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$
Alt.K-L/r	1.0	1.1	1.1	1.3	1.3	1.5	1.3	1.5
1	50	51	51	44	49	44	46	51
2	34	37	34	29	32	29	33	36
5	19	23	19	17	19	17	21	22
15	11	9	10	11	10	11	9	9
17	78	72	78	75	78	77	76	78
20	17	15	17	16	16	15	15	15
22	70	62	70	69	69	65	66	64
25	34	30	33	33	33	31	31	30
27	44	40	43	42	42	40	42	41
28	43	36	42	42	41	41	37	37
Av.	40,1	37,4	39,8	37,9	38,9	37,1	37,5	38,3

Table 5: Powers of 5% tests under symmetric alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 37.

$n=20$										
skew										
k	1				2					
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_3}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_3}^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$
Alt.K-L/r	0.5	1.5	1.7	2.0	0.1	0.1	0.3	1.0	1.0	1.1
36	59	38	35	30	34	32	26	38	56	33
40	30	25	25	26	36	29	31	37	31	35
41	27	25	26	28	33	29	31	33	27	32
44	16	16	16	17	23	19	22	23	17	23
45	27	29	30	32	33	32	35	31	28	33
46	26	26	27	29	35	31	34	34	28	34
47	21	22	23	24	24	24	26	21	22	23
48	78	79	79	80	72	78	74	67	71	71
49	23	24	25	26	27	27	28	25	25	27
50	53	51	52	53	61	56	58	62	55	61
52	57	58	59	59	51	58	56	46	54	51
54	66	66	66	66	60	65	64	51	61	60
57	71	72	72	70	63	66	56	65	69	62
58	92	90	90	90	90	92	90	85	91	89
Av.	46.2	44.2	44.7	45.0	45.8	45.6	45.1	44.2	45.3	45.3
k	2						3		4	5
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$
Alt.K-L/r	1.1	1.3	1.3	1.5	1.7	2.0	0.1	0.3	0.5	0.7
36	54	26	49	43	38	29	51	57	63	68
40	32	31	31	31	29	25	31	41	40	40
41	29	31	31	32	33	31	26	34	31	29
44	18	22	19	20	19	19	16	22	20	20
45	30	35	32	34	35	36	25	28	24	23
46	29	34	30	30	31	30	26	32	29	29
47	23	25	23	25	26	27	17	18	16	16
48	75	74	78	81	82	81	73	68	65	64
49	25	28	26	27	28	28	21	23	20	20
50	56	57	56	56	54	51	56	63	61	62
52	56	57	58	59	60	61	51	44	42	42
54	64	64	66	66	66	66	65	58	59	59
57	69	56	69	67	64	54	74	64	58	54
58	92	90	92	91	90	88	94	95	95	96
Av.	46.5	45.0	47.1	47.3	46.7	44.7	44.7	46.3	44.5	44.4

Table 6: Powers of 5% tests under skew alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 44.

$n=50$								
symmetric								
k	1					2		
Tests	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$
Alt.M-S/r	0.3	0.3	0.9	1.0	1.1	0.3	0.5	0.5
1	88	95	92	93	85	93	94	95
4	59	67	60	62	56	77	72	59
7	21	23	19	20	19	38	31	16
15	24	25	25	25	24	18	19	24
18	57	59	57	58	57	53	54	61
21	61	62	61	62	62	54	52	59
24	89	89	89	89	89	86	85	88
27	78	78	78	78	78	76	75	77
30	53	55	53	54	53	47	46	53
32	100	100	100	100	100	100	100	100
Av.	62.9	65.5	63.3	64.0	62.3	64.0	62.8	63.2
k	2				3			
Tests	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$
Alt.M-S/r	0.7	0.7	0.9	1.0	0.9	1.0	1.3	1.5
1	94	93	95	94	95	95	93	94
4	63	53	62	59	72	71	61	62
7	21	14	20	19	32	31	23	23
15	25	24	25	25	18	18	21	21
18	63	61	64	63	56	56	64	65
21	58	59	58	58	49	48	52	51
24	87	88	87	87	84	83	85	84
27	76	77	76	76	73	72	74	73
30	54	53	54	54	45	45	50	50
32	100	100	100	99	100	100	100	100
Av.	64.2	62.2	64.3	63.4	62.4	62.0	62.2	62.2

Table 7: Powers of 5% tests under symmetric alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 62.

$n = 50$									
skew									
k	2						3		
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt.M-S/r	0.7	0.9	1.0	1.1	1.3	1.5	0.1	0.3	1.3
36	98	98	98	97	95	92	94	84	99
38	68	65	64	63	64	66	70	78	63
40	71	74	73	73	69	65	80	71	77
42	93	100	100	100	100	100	100	99	98
44	36	40	40	40	39	38	45	38	37
50	92	93	93	93	92	90	95	93	91
52	87	88	88	88	89	89	85	84	79
54	92	92	92	93	93	93	92	92	91
56	100	100	100	100	100	100	100	100	100
58	100	100	100	100	100	100	100	100	100
Av.	83.8	84.9	84.8	84.7	84.0	83.2	86.2	84.0	83.6
k	3			4			5		
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt.M-S/r	1.5	1.7	2.0	0.1	0.3	0.5	0.1	0.3	0.5
36	99	98	97	99	97	93	99	99	98
38	61	60	60	71	74	73	71	71	70
40	78	77	74	82	79	73	82	82	78
42	100	100	100	100	100	100	100	100	100
44	40	41	40	44	41	36	41	43	39
50	92	92	92	96	95	93	95	96	94
52	79	80	80	83	81	80	80	80	78
54	91	91	91	92	92	92	92	92	91
56	100	100	100	100	100	100	100	100	100
58	100	100	100	100	100	100	100	100	100
Av.	84.0	83.9	83.3	86.6	85.9	83.9	86.0	86.2	84.7

Table 8: Powers of 5% tests under skew alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 83.

$n=50$										
symmetric										
k	1			2						
Tests	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n;c_2}$	$\hat{T}^{(r,k)}_{n;c_2}$	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n}$	$\hat{T}^{(r,k)}_{n;c_2}$	$\hat{T}^{(r,k)}_{n}$	$\hat{T}^{(r,k)}_{n;c_2}$	$\hat{T}^{(r,k)}_{n;c_2}$
Alt.K-L/r	0.3	0.9	1.0	0.3	0.5	0.5	0.7	0.7	0.9	1.0
1	95	92	93	93	94	95	94	93	95	94
2	83	79	80	87	86	80	81	76	82	79
5	57	51	53	70	64	48	52	43	52	50
15	25	24	25	18	19	24	24	24	24	25
17	91	87	88	98	97	97	95	96	94	92
20	42	42	42	36	34	39	38	39	38	38
22	98	98	98	97	97	98	98	98	97	96
25	69	68	68	64	63	67	65	66	66	65
27	78	78	78	75	75	77	76	77	76	76
28	82	80	81	78	77	82	83	82	83	82
Av.	72.0	69.8	70.6	71.7	70.5	70.8	70.7	69.4	70.7	69.8

k	3								
Tests	$\hat{T}^{(r,k)}_{n;c_3}$	$\hat{T}^{(r,k)}_{n}$	$\hat{T}^{(r,k)}_{n;c_1}$	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n}$	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n}$	$\hat{T}^{(r,k)}_{n;c_4}$	$\hat{T}^{(r,k)}_{n}$
Alt.K-L/r	-0.499	-0.499	0.5	0.9	0.9	1.0	1.0	1.1	1.1
1	90	90	90	95	95	96	95	95	95
2	86	86	86	85	82	85	81	85	80
5	70	70	70	63	54	63	53	62	52
15	15	15	15	18	20	18	20	18	20
17	98	98	98	99	99	99	99	99	99
20	30	30	30	31	33	30	33	29	32
22	96	96	96	97	97	97	97	96	97
25	60	60	60	61	62	60	62	59	62
27	73	73	73	73	74	72	74	72	74
28	73	73	73	78	79	77	79	77	80
Av.	69.1	69.1	69.0	69.9	69.6	69.7	69.4	69.1	69.1

Table 9: Powers of 5% tests under symmetric alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 69.

$n=50$								
skew								
k	1				2			
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_3}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt.K-L/r	0.5	1.3	1.5	1.7	0.1	0.7	0.9	1.0
36	97	92	87	80	62	99	98	98
40	75	65	66	65	72	72	74	74
41	62	56	60	62	69	52	60	63
44	41	36	38	39	48	37	40	41
45	58	58	60	63	64	55	57	58
46	60	57	59	60	68	56	60	60
47	45	46	46	48	41	42	42	43
48	99	99	99	99	97	97	98	98
49	47	47	48	50	49	46	47	48
50	93	91	91	92	94	92	93	93
52	90	90	90	90	79	87	88	88
54	93	93	93	93	88	92	92	93
57	97	97	97	97	96	97	97	97
58	100	100	100	100	97	100	100	100
Av.	75.5	73.5	74.0	74.1	73.1	73.2	74.7	75.2
k	2				3		4	
Tests	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt.K-L/r	1.1	1.3	1.5	1.7	0.1	2.0	0.1	0.3
36	97	95	92	88	93	97	99	97
40	73	70	65	60	80	74	82	79
41	65	68	69	68	66	70	61	62
44	41	40	38	36	45	40	44	41
45	59	62	64	66	57	55	51	51
46	61	61	59	58	63	58	61	60
47	43	45	46	47	36	34	32	31
48	98	99	99	99	98	97	97	97
49	48	50	50	51	45	42	41	41
50	93	92	90	89	95	92	96	95
52	88	89	89	89	85	81	83	81
54	93	93	93	93	92	91	92	92
57	98	98	97	97	98	97	98	97
58	100	100	100	100	100	100	100	100
Av.	75.4	75.6	75.0	74.4	75.3	73.4	74.1	73.2

Table 10: Powers of 5% tests under skew alternatives based on 100 000 simulations using empirical critical values with Av. ≥ 73.

Heaving a look at the above simulations we see that our most powerful tests for $n=20$ are $\hat{T}_{n;c_4}^{(0.3,1)}$ with Av.=36.0 under symmetric Alt.M-S while the standard test K^2 has Av.=36.3 (Table 1); $\hat{T}_{n;c_2}^{(0.5,4)}$ with Av.=58.2 under skew Alt.M-S while the standard test S-W has Av.=59.6 (Table 1); $\hat{T}_n^{(1.0,3)}$ with Av.=40.1 under symmetric Alt.K-L while the standard test W has Av.=39.0 (Table 2); $\hat{T}_n^{(1.5,2)}$ with Av.=47.3 under skew Alt.K-L while the standard test W has Av.=46.9 (Table 2).

The most powerful tests for $n=50$ are: $\hat{T}_{n;c_4}^{(0.3,1)}$ with Av.=65.5 under symmetric Alt.M-S while the standard test K^2 has Av.=61.6 (Table 1); $\hat{T}_n^{(0.1,4)}$ with Av.=86.6 under skew Alt.M-S while the standard test S-W has Av.=88.3 (Table 1); $\hat{T}_{n;c_4}^{(0.3,1)}$ with Av.=72.0 under symmetric Alt.K-L while the standard test W has Av.=69.5 (Table 2); $\hat{T}_n^{(1.3,2)}$ with Av.=75.6 under skew Alt.K-L while the standard test W has Av.=73.1 (Table 2).

Hence we conclude that our tests for normality perform very well and they can be recommended to use them in the statistical inference. Moreover, our simulations show that tests K^2, R and Y are poor when alternatives are skew. For skew alternatives the test S-W and ours are best.

$n\backslash$Alt.	M-S symmetric	M-S skew	K-L symmetric	K-L skew
20	$\hat{T}_{20;c_4}^{(0.3,1)}$ Av.=36.0	$\hat{T}_{20;c_2}^{(0.5,4)}$ Av.=58.2	$\hat{T}_{20}^{(1.0,3)}$ Av.=40.1	$\hat{T}_{20}^{(1.5,2)}$ Av.=47.3
50	$\hat{T}_{50;c_4}^{(0.3,1)}$ Av.=65.5	$\hat{T}_{50}^{(0.1,4)}$ Av.=86.6	$\hat{T}_{50;c_4}^{(0.3,1)}$ Av.=72.0	$\hat{T}_{50}^{(1.5,2)}$ Av.=75.0

Table 11: Our simple recommended tests.

It is almost sure that one can find in our family tests better than those presented here choosing different values of r and k.

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