

**BERNOULLI POLYNOMIALS,
FOURIER SERIES AND ZETA NUMBERS**

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Abstract: Fourier series for Bernoulli polynomials are used to obtain information about values of the Riemann zeta function for integer arguments greater than one. If the argument is even we recover the well-known exact values, if the argument is odd we find integral representations and rapidly convergent series.

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1. Bernoulli Polynomials and Fourier Series

The Bernoulli polynomials $B_n(x)$ are for $x \in \mathbb{R}$ defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (1)$$

The polynomials have the form

$$B_n(x) = \sum_{j=0}^n \binom{n}{j} B_j x^{n-j}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0, \quad (2)$$

where B_0, B_1, B_2, \dots are the Bernoulli numbers defined by (1) with $x = 0$, i.e.

$B_n = B_n(0)$. The first Bernoulli polynomial is $B_0(x) = 1$, and the next three are as follows,

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

The periodic Bernoulli functions $\tilde{B}_n(x)$ has period 1, and is for $n \in \mathbb{N}$ defined by

$$\tilde{B}_n(x) = B_n(x), \quad 0 \leq x < 1, \quad \tilde{B}_n(x+1) = \tilde{B}_n(x), \quad x \in \mathbb{R}. \quad (3)$$

The functions $\tilde{B}_n(x)$ are continuous with continuous derivatives up to order $n-1$ if $n \geq 2$. Fourier series for the periodic Bernoulli functions are for even index

$$\tilde{B}_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{\cos(2m\pi x)}{m^{2n}}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \quad (4)$$

and for odd index

$$\tilde{B}_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{m=1}^{\infty} \frac{\sin(2m\pi x)}{m^{2n+1}}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (5)$$

This Fourier series can be found in for example [5] or [7], where they are defined for $B_{2n}(x)$ and $B_{2n+1}(x)$ restricted to $x \in [0, 1]$.

The aim of this paper is to apply the Fourier series in (5) for finding new integral representations and rapidly convergent series for values of the Riemann zeta function, when the argument is an odd integer greater than one, specially if the argument is 3. In [3] are some examples where Fourier series is applied to find values of the Riemann zeta function, when the argument is an even integer. In the next section the general formula for the values of the Riemann zeta function, when the argument is an even integer, is obtained from the Fourier series (4). In [8] Fourier series is applied to generate some special integrals, sums of series, and combinatorial identities.

2. Zeta Numbers

The Riemann zeta function is for $\Re s > 1$ defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad (6)$$

and the zeta numbers is values of $\zeta(s)$ when the argument $s \in \mathbb{N} \setminus \{1\}$, see [5]. Setting $x = 0$ in (4) we get the well-known formula

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad n \in \mathbb{N}, \quad (7)$$

where B_2, B_4, B_6, \dots are Bernoulli numbers, and we have used that $\tilde{B}_{2n}(0) = B_{2n}(0) = B_{2n}$. Setting x to a particular value in (5) will not give us any information about $\zeta(2n + 1)$, $n \in \mathbb{N}$, but using the well-known formula

$$\int_0^\infty \frac{\sin(ux)}{x} dx = \frac{\pi}{2}, \quad u > 0, \quad (8)$$

and the uniform convergens of the series for $\tilde{B}_{2n+1}(x)/x$ obtained from (5), we get for $n \in \mathbb{N}$:

$$\begin{aligned} \int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx &= (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{m=1}^\infty \frac{1}{m^{2n+1}} \cdot \frac{\pi}{2} \\ &= (-1)^{n+1} \frac{(2n+1)!}{2(2\pi)^{2n}} \zeta(2n+1). \end{aligned}$$

Now we have the following information of zeta numbers with odd argument

$$\zeta(2n+1) = (-1)^{n+1} \frac{2(2\pi)^{2n}}{(2n+1)!} \int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx, \quad n \in \mathbb{N}. \quad (9)$$

Using that $\tilde{B}_{2n+1}(x+1) = \tilde{B}_{2n+1}(x)$ and $\tilde{B}_{2n+1}(-x) = -\tilde{B}_{2n+1}(x)$ then for $n \in \mathbb{N}$ we get

$$\begin{aligned} \int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx &= \int_0^1 \frac{B_{2n+1}(x)}{x} dx + \sum_{k=1}^\infty \int_{2k-1}^{2k+1} \frac{\tilde{B}_{2n+1}(x)}{x} dx \\ &= \int_0^1 \frac{B_{2n+1}(x)}{x} dx + \sum_{k=1}^\infty \int_{-1}^1 \frac{\tilde{B}_{2n+1}(x)}{x+2k} dx, \end{aligned}$$

where

$$\begin{aligned} \int_{-1}^1 \frac{\tilde{B}_{2n+1}(x)}{x+2k} dx &= \int_{-1}^0 \frac{\tilde{B}_{2n+1}(x)}{x+2k} dx + \int_0^1 \frac{\tilde{B}_{2n+1}(x)}{x+2k} dx \\ &= \int_0^1 \frac{B_{2n+1}(x)}{x-2k} dx + \int_0^1 \frac{B_{2n+1}(x)}{x+2k} dx. \end{aligned}$$

Consequently we have

$$\int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx = \int_0^1 \frac{B_{2n+1}(x)}{x} dx + \sum_{k=1}^\infty \int_0^1 \frac{2xB_{2n+1}(x)}{x^2 - 4k^2} dx, \quad n \in \mathbb{N}. \quad (10)$$

From (9) and (10) we can go two ways, one way to get an integral representation for $\zeta(2n+1)$, and another way to get a series representation for $\zeta(2n+1)$.

3. Integral representations of Zeta(2n+1)

Changing the order of summation and integration in (10), and using that

$$\sum_{k=1}^\infty \frac{2x}{x^2 - 4k^2} = \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{1}{x}, \quad x \in \mathbb{R} \setminus \{0, \pm 2, \pm 4, \dots\},$$

we get for $n \in \mathbb{N}$

$$\int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx = \int_0^1 \frac{B_{2n+1}(x)}{x} dx + \int_0^1 \left(\sum_{k=1}^\infty \frac{2xB_{2n+1}(x)}{x^2 - 4k^2} \right) dx,$$

where

$$\begin{aligned} \int_0^1 \left(\sum_{k=1}^\infty \frac{2xB_{2n+1}(x)}{x^2 - 4k^2} \right) dx &= \int_0^1 B_{2n+1}(x) \left(\sum_{k=1}^\infty \frac{2x}{x^2 - 4k^2} \right) dx \\ &= \int_0^1 B_{2n+1}(x) \left(\frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{1}{x} \right) dx. \end{aligned}$$

Consequently we have

$$\int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx = \frac{\pi}{2} \int_0^1 B_{2n+1}(x) \cot\left(\frac{\pi x}{2}\right) dx, \quad n \in \mathbb{N}. \quad (11)$$

Inserting (11) in (9) we get the integral representation

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot\left(\frac{\pi x}{2}\right) dx, \quad n \in \mathbb{N}. \quad (12)$$

In [2] we have for $\delta = 1$ or $\delta = \frac{1}{2}$ the integral representations

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2\delta(2n+1)!} \int_0^\delta B_{2n+1}(x) \cot(\pi x) dx, \quad n \in \mathbb{N}. \quad (13)$$

Because

$$\cot\left(\frac{\pi x}{2}\right) - \cot(\pi x) = \frac{\sin(\pi x)}{1 - \cos(\pi x)} - \frac{\cos(\pi x)}{\sin(\pi x)} = \frac{1}{\sin(\pi x)}, \quad x \notin \mathbb{Z},$$

we get for $n \in \mathbb{N}$:

$$\int_0^1 B_{2n+1}(x) \cot\left(\frac{\pi x}{2}\right) dx - \int_0^1 B_{2n+1}(x) \cot(\pi x) dx = \int_0^1 \frac{B_{2n+1}(x)}{\sin(\pi x)} dx. \quad (14)$$

Inserting the Fourier series (5) we get

$$\int_0^1 \frac{B_{2n+1}(x)}{\sin(\pi x)} dx = 0, \quad n \in \mathbb{N}, \quad (15)$$

because

$$\int_0^1 \frac{\sin(2m\pi x)}{\sin(\pi x)} dx = 0, \quad m \in \mathbb{N}.$$

From(14) and (15) it follows that (12) is equal to (13) if $\delta = 1$.

Integration of $\tilde{B}_{2n+1}(x)/x$ can be done in an alternative way than we have seen in the previous. This will lead us to an integral representation which involve the Gamma function. The alternative way is

$$\begin{aligned} \int_0^\infty \frac{\tilde{B}_{2n+1}(x)}{x} dx &= \sum_{k=0}^\infty \int_k^{k+1} \frac{\tilde{B}_{2n+1}(x)}{x} dx \\ &= \sum_{k=0}^\infty \int_0^1 \frac{B_{2n+1}(x)}{x+k} dx. \end{aligned}$$

Further we have

$$\begin{aligned} \sum_{k=0}^\infty \int_0^1 \frac{B_{2n+1}(x)}{x+k} dx &= - \sum_{k=0}^\infty \int_0^1 \left(\frac{1}{k+1} - \frac{1}{x+k} \right) B_{2n+1}(x) dx = \\ &- \int_0^1 \sum_{k=0}^\infty \left(\frac{1}{k+1} - \frac{1}{x+k} \right) B_{2n+1}(x) dx \\ &- \int_0^1 \left\{ -\gamma + \sum_{k=0}^\infty \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \right\} B_{2n+1}(x) dx, \end{aligned}$$

where we have changed the order of summation and integration in the second line, and the Euler constant γ is subtracted from the sum in the third line. Two times we have used that

$$\int_0^1 B_{2n+1}(x)dx = 0.$$

The digamma function is defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{z+k} \right), \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

and therefore

$$\int_0^{\infty} \frac{\tilde{B}_{2n+1}(x)}{x} dx = - \int_0^1 \psi(x) B_{2n+1}(x) dx, \quad n \in \mathbb{N}. \quad (16)$$

Using partial integration on the integral in (16) we get

$$\int_0^{\infty} \frac{\tilde{B}_{2n+1}(x)}{x} dx = (2n+1) \int_0^1 B_{2n}(x) \ln \Gamma(x) dx, \quad n \in \mathbb{N}, \quad (17)$$

where we have used that $B'_{2n+1}(x) = (2n+1)B_{2n}(x)$, and

$$\lim_{t \rightarrow 1^-} (\ln \Gamma(x) B_{2n+1}(x)) = \lim_{t \rightarrow 0^+} (\ln \Gamma(x) B_{2n+1}(x)) = 0.$$

Inserting (17) in (9) we get

$$\zeta(2n+1) = (-1)^{n+1} \frac{2(2\pi)^{2n}}{(2n)!} \int_0^1 B_{2n}(x) \ln \Gamma(x) dx, \quad n \in \mathbb{N}. \quad (18)$$

Compared with (12) and (13) we see that involving the gamma function, we get $B_{2n}(x)$ instead of $B_{2n+1}(x)$ in the integrand.

4. Series Representations of Zeta(2n+1)

From (2) we can find formulas for the integrals on the righthand side of (10). For the first integral we find

$$\int_0^1 \frac{B_{2n+1}(x)}{x} dx = \int_0^1 \sum_{j=0}^{2n} \binom{2n+1}{j} B_j x^{2n-j} dx$$

$$= \sum_{j=0}^{2n} \binom{2n+1}{j} B_j \int_0^1 x^{2n-j} dx,$$

i.e.

$$\int_0^1 \frac{B_{2n+1}(x)}{x} dx = \sum_{j=0}^{2n} \binom{2n+1}{j} \frac{B_j}{2n+1-j}. \tag{19}$$

For the integrals in the sum we find

$$\int_0^1 \frac{2xB_{2n+1}(x)}{x^2 - 4k^2} dx = \int_0^1 \left(p_{2n}(x, k) + \frac{B_{2n+1}(2k)}{x - 2k} + \frac{B_{2n+1}(-2k)}{x + 2k} \right) dx,$$

where $p_{2n}(x, k)$ is a polynomial of degree $2n$ in x and k , fx. we have $p_2(x, k) = 2x^2 - 3x + 8k^2 + 1$. From $B_n(-x) = (-1)^n [B_n(x) + nx^{n-1}]$, we get $B_{2n+1}(-2k) = -[B_{2n+1}(2k) + (2n+1)(2k)^{2n}]$. Then it follows that

$$\begin{aligned} & \int_0^1 \frac{2xB_{2n+1}(x)}{x^2 - 4k^2} dx = \\ & \int_0^1 \left(p_{2n}(x, k) + \frac{B_{2n+1}(2k)}{x - 2k} - \frac{B_{2n+1}(2k)}{x + 2k} - \frac{(2n+1)(2k)^{2n}}{x + 2k} \right) dx = \\ & P_{2n}(k) - B_{2n+1}(2k) \ln \frac{2k+1}{2k-1} - (2n+1)(2k)^{2n} \ln \frac{2k+1}{2k}, \end{aligned} \tag{20}$$

where $P_{2n}(k)$ is a polynomial of degree $2n$ in k , fx. we have $P_2(k) = 8k^2 + \frac{1}{6}$. Substituting (19) and (20) in (10), we get from (9)

$$\begin{aligned} \zeta(2n+1) &= (-1)^{n+1} \frac{2(2\pi)^{2n}}{(2n+1)!} \left[\sum_{j=0}^{2n} \binom{2n+1}{j} \frac{B_j}{2n+1-j} \right. \\ & \left. + \sum_{k=1}^{\infty} \left(P_{2n}(k) - B_{2n+1}(2k) \ln \frac{2k+1}{2k-1} - (2n+1)(2k)^{2n} \ln \frac{2k+1}{2k} \right) \right]. \end{aligned} \tag{21}$$

5. Series Representations of Zeta(3)

Setting $n = 1$ in (21), we get

$$\zeta(3) = \frac{2(2\pi)^2}{3!} \left[\frac{B_0}{3} + \frac{3B_1}{2} + 3B_2 \right]$$

$$+ \sum_{k=1}^{\infty} \left(8k^2 + \frac{1}{6} - (8k^3 - 6k^2 + k) \ln \frac{2k+1}{2k-1} - 12k^2 \ln \frac{2k+1}{2k} \right) \Big].$$

Inserting the Bernoulli numbers B_0, B_1, B_2 , and using that

$$\begin{aligned} 6k^2 \ln \frac{2k+1}{2k-1} - 12k^2 \ln \frac{2k+1}{2k} &= 6k^2 \ln \left(\frac{2k+1}{2k-1} \cdot \frac{4k^2}{(2k+1)^2} \right) \\ &= -6k^2 \ln \left(1 - \frac{1}{4k^2} \right), \end{aligned}$$

we get

$$\begin{aligned} \zeta(3) &= \\ \frac{4\pi^2}{3} &\left[\frac{1}{12} + \sum_{k=1}^{\infty} \left(8k^2 + \frac{1}{6} - 6k^2 \ln \left(1 - \frac{1}{4k^2} \right) - (8k^3 + k) \ln \frac{2k+1}{2k-1} \right) \right]. \quad (22) \end{aligned}$$

Using the fact that

$$-\ln \left(1 - \frac{1}{4k^2} \right) = \sum_{m=0}^{\infty} \frac{1}{m+1} \left(\frac{1}{4k^2} \right)^{m+1},$$

and

$$\ln \frac{2k+1}{2k-1} = \ln \frac{1 + \frac{1}{2k}}{1 - \frac{1}{2k}} = \sum_{m=0}^{\infty} \frac{2}{2m+1} \left(\frac{1}{2k} \right)^{2m+1},$$

we get

$$\begin{aligned} &\sum_{k=1}^{\infty} \left(8k^2 + \frac{1}{6} - 6k^2 \ln \left(1 - \frac{1}{4k^2} \right) - (8k^3 + k) \ln \frac{2k+1}{2k-1} \right) \\ &= \sum_{k=1}^{\infty} \left(8k^2 + \frac{1}{6} + \sum_{m=0}^{\infty} \frac{3}{2m+2} \frac{1}{4^m k^{2m}} - \sum_{m=0}^{\infty} \frac{1}{2m+1} \frac{8k^2+1}{4^m k^{2m}} \right), \end{aligned}$$

where the parantes back the summation can be written

$$\begin{aligned} &8k^2 + \frac{1}{6} + \frac{3}{2} - 8k^2 - 1 + \sum_{m=1}^{\infty} \left(\frac{3}{2m+2} \frac{1}{4^m k^{2m}} - \frac{1}{2m+1} \frac{8k^2+1}{4^m k^{2m}} \right) \\ &= \frac{2}{3} + \sum_{m=1}^{\infty} \left(\frac{3}{2m+2} - \frac{1}{2m+1} \right) \frac{1}{4^m k^{2m}} - \sum_{m=1}^{\infty} \frac{2}{2m+1} \frac{1}{4^{m-1} k^{2(m-1)}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{3} + \sum_{m=1}^{\infty} \left(\frac{3}{2m+2} - \frac{1}{2m+1} \right) \frac{1}{4^m k^{2m}} - \frac{2}{3} - \sum_{m=2}^{\infty} \frac{2}{2m+1} \frac{1}{4^{m-1} k^{2(m-1)}} \\
 &= \sum_{m=1}^{\infty} \left(-\frac{1}{2m+1} + \frac{3}{2m+2} - \frac{2}{2m+3} \right) \frac{1}{4^m k^{2m}}.
 \end{aligned}$$

Inserting in (22) gives

$$\begin{aligned}
 \zeta(3) &= \frac{4\pi^2}{3} \left[\frac{1}{12} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(-\frac{1}{2m+1} + \frac{3}{2m+2} - \frac{2}{2m+3} \right) \frac{1}{4^m k^{2m}} \right] \\
 &= \frac{4\pi^2}{3} \left[\frac{1}{12} - \sum_{m=1}^{\infty} \left(\frac{1}{2m+1} - \frac{3}{2m+2} + \frac{2}{2m+3} \right) \frac{1}{4^m} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} \right],
 \end{aligned}$$

i.e.

$$\zeta(3) = \frac{4\pi^2}{3} \left[\frac{1}{12} - \sum_{m=1}^{\infty} \left(\frac{1}{2m+1} - \frac{3}{2m+2} + \frac{2}{2m+3} \right) \frac{\zeta(2m)}{4^m} \right]. \tag{23}$$

Using that $\zeta(0) = -\frac{1}{2}$ we get from (20)

$$\zeta(3) = -\frac{4\pi^2}{3} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} - \frac{3}{2m+2} + \frac{2}{2m+3} \right) \frac{\zeta(2m)}{4^m}. \tag{24}$$

Reducing the parantes we get

$$\zeta(3) = \frac{4\pi^2}{3} \sum_{m=0}^{\infty} \frac{2m-1}{(2m+1)(2m+2)(2m+3)} \frac{\zeta(2m)}{4^m}. \tag{25}$$

Series representation analogous to (25) can be found in other papers, see for example [1], [3], [4] and [6], but I have not found exactly the representation (25). Here is some of the other series representations

$$\zeta(3) = -\frac{8\pi^2}{9} \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+3)} \frac{\zeta(2m)}{4^m}, \tag{26}$$

and

$$\zeta(3) = -\frac{2\pi^2}{7} \sum_{m=0}^{\infty} \frac{1}{(m+1)(2m+1)} \frac{\zeta(2m)}{4^m}. \tag{27}$$

6. Computing Zeta(3) from Rapidly Convergent Series

An approximately value for $\zeta(3)$ using (25) is for $M \in \mathbb{N}$ given by

$$\zeta(3) \cong \frac{4\pi^2}{3} \sum_{m=0}^M \frac{2m-1}{(2m+1)(2m+2)(2m+3)} \frac{\zeta(2m)}{4^m} \quad (28)$$

with an error R_M satisfying

$$|R_M| < \frac{4\pi^2}{3} \frac{(2M+1)\zeta(2M+2)}{(2M+3)(2M+4)(2M+5)} \sum_{m=M+1}^{\infty} \frac{1}{4^m}.$$

If we use that $\zeta(s) < (1 - 2^{1-s})^{-1}$, $s > 1$, and

$$\sum_{m=M+1}^{\infty} \frac{1}{4^m} = \frac{1}{3 \cdot 4^M},$$

we get

$$|R_M| < \frac{4\pi^2}{9} \frac{(2M+1)}{(2M+3)(2M+4)(2M+5)(4^M - \frac{1}{2})}. \quad (29)$$

For $M = 25$ we get from (28) and (29) the approximately zeta value

$$\zeta(3) = 1.20205\ 69031\ 59594\ 284$$

with an error bound $|R_{25}| < 2 \cdot 10^{-18}$.

7. Conclusion

Fourier series for even-indexed Bernoulli polynomials are leading to well-known explicit formulas for the zeta numbers $\zeta(2n)$, $n \in \mathbb{N}$. Fourier series for odd-indexed Bernoulli polynomials do not lead to analogous closed formulas for the zeta numbers $\zeta(2n+1)$, $n \in \mathbb{N}$. Using Fourier series for periodic Bernoulli functions $\tilde{B}_{2n+1}(x)$ and integration of the series for $\tilde{B}_{2n+1}(x)/x$ term by term leads to integral representations and series representations of $\zeta(2n+1)$, $n \in \mathbb{N}$. The series representations are used to evaluate rapidly convergent series for $\zeta(3)$.

References

- [1] Raymond Ayoub, Euler and the zeta function, *Amer. Math. Monthly*, **81** (1974), 1067-1086.
- [2] D. Cvijovic, J. Klinowski, Integral representation of the Riemann zeta function for odd-integer arguments, *J. Comput. Appl. Math.*, **142** (2002), 435-439, doi: 10.1016/S0377-0427(02)00358-8.
- [3] E.E. Scheufens, From Fourier series to rapidly convergent series for Zeta(3), *Math. Mag.*, **84** (2011), 26-32.
- [4] E.E. Scheufens, Euler polynomials, Fourier series and zeta numbers, *Int. J. Pure Appl. Math.*, **78** (2012), 37-47.
- [5] J. Spanier, K.B. Oldham, *An Atlas of Functions*, Springer, Berlin, 1987.
- [6] H.M. Srivastava, Some families of rapidly convergent series representations for the zeta function, *Taiwanese J. Math.*, **4** (2000), 569-599, http://www.math.nthu.edu.tw/~tjm/abstract/0012/tjm0012_3.pdf.
- [7] Nico M. Temme, *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.
- [8] R. Witula, D. Jama, D. Slota, P. Zwolenski, Some Fourier series expansions and their applications, *Int. J. Pure Appl. Math.*, **64** (2010), 199-223.

