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WALSH-FOURIER COEFFICIENTS OF LINEAR MAPPINGS

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Abstract: Let (x_n) be a sequence of elements of a locally convex space X. Let $H(\mathcal{C})$ be a homogeneous Banach space on the Cantor group $\mathcal{C}$. In this paper the necessary and sufficient conditions are given for (x_n) to be the Walsh-Fourier coefficients of some continuous, weakly compact or compact linear mapping $u : H(\mathcal{C}) \to X$.

(As for Cantor group and related facts, see R.E. Edwards, *Fourier Series*, at the end of the paper.)

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1. The Walsh-Fourier Coefficients

Let $\mathcal{C}$ be the Cantor group, R and Z denoting the additive group of reals, integers, respectively. Let $H(\mathcal{C})$ be a homogeneous Banach space on the Cantor group $\mathcal{C}$ with the norm $\|\|_{L^1} \leq \|\|_H$. Let X be a quasi-complete locally convex (Hausdorff) topological vector space and $u : H(\mathcal{C}) \to X$ a continuous linear mapping. The Walsh-Fourier coefficients of the mapping u are, by definition, the elements of X of the form

$$\hat{u}(n) = u(w_n),\, n \in Z,$$

w_n denoting the Walsh function. Let (x_n) be a sequence of elements of X. In this paper the necessary and sufficient conditions are given for (x_n) to be

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the Walsh-Fourier coefficients of some continuous, weakly compact or compact linear mapping $u : H(\mathcal{C}) \to X$, in particular if $H(\mathcal{C}) = C(\mathcal{C})$, to be the Walsh-Fourier-Stieltjes coefficients of a regular vector measure on $\mathcal{C}$ with values in X (see [2], [9] and [13]). The results are a generalization of the results of ([8], p. 34 and ff.) proved for a sequence of complex numbers.

1. Recall that $H(\mathcal{C})$ is a linear subspace of the Banach space $L^1(\mathcal{C})$ (of all complex-valued Lebesgue integrable functions on $\mathcal{C}$ having a norm $\|\|_H \geq \|\|_{L^1}$, under which it is a Banach space having the properties:

(1) If $f \in H(\mathcal{C})$ and $v \in \mathcal{C}$, then $f_v \in H(\mathcal{C})$ and $\|f_v\|_H = \|f\|_H$

$$\{f_v(t) = f(t - v)\}$$

(2) for all $f \in H(\mathcal{C})$ $v, v_0 \in \mathcal{C}$,$\lim_{v \to v_0} \|f_v - f_{v_0}\|_H = 0$.

Examples of homogeneous Banach spaces on $\mathcal{C}$ are (cf. [8]): the space $C(\mathcal{C})$ of all continuous functions, the space $C^n(\mathcal{C})$ of all n-times continuously differentiable functions, the spaces $L^p(\mathcal{C}), 1 \leq p < \infty$.

A Walsh polynomial on $\mathcal{C}$ is a function $a = a(g)$ defined on $\mathcal{C}$ by $a(g) = \sum_0^n a_j w_j(g)$.

Denote by $p(\mathcal{C})$ the set of all Walsh polynomials on $\mathcal{C}$. We shall need the following theorem ([8], Th. 2.12).

Theorem 1. *For every $f \in H(\mathcal{C})$ we have $\sigma_n(f) \to f$, $n \to \infty$, in the $H(\mathcal{C})$ norm.*

Recall that

$$\sigma_n(f, g) = \sum_{-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) \hat{f}(j) w_j(g),$$

where $\hat{f}(j)$ is the jth Fourier—Lebesgue coefficient of f defined by

$$\hat{f}(j) = \frac{1}{2\pi} \int f(g) w_j(g) dg.$$

Let the locally convex topology of the space X be defined by a family $Q = (q)$ of continuous seminorms. For a continuous seminorm q and for a linear mapping u from $H(\mathcal{C})$ into X we denote

$$\|u\|_q = \sup\big\{q(u(f)), f \in H(\mathbf{C}), \|f\|_H \leq 1\big\}.$$

Lemma 1. *Let $u : H(\mathcal{C}) \to X$ be a continuous linear mapping. For every $a = \sum_0^n a_j w_j(g)$ we have $u(a) = \sum_0^n a_j \hat{u}(j)$ and $q(u(a)) \leq \|a\|_H \|u\|_q$ for every continuous seminorm q.*

Theorem 2. (Parseval's formula) *Let $f \in H(\mathcal{C})$ and $u : H(\mathcal{C}) \to X$ be a continuous linear mapping. Then*

$$u(f) = \lim_{N\to\infty} \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) \hat{f}(j)\hat{u}(j).$$

Proof. Since, by Theorem 1, $f = \lim_{n\to\infty} \sigma_n(f)$ in the $H(\mathcal{C})$ norm, it follows from Lemma 1 and the continuity of u that the assertion is true. □

Theorem 3. *Let (x_j) be a sequence of elements of X. Then the following two conditions are equivalent:*

(a) *There is a continuous linear mapping $u : H(\mathcal{C}) \to X$, with $\|u\|_q \le C_q < \infty$ for every continuous seminorm q, such that $\hat{u}(j) = x_j$ for all $j \in Z$.*

(b)

For all Walsh polynomials $a = \sum_{j=0}^{l} a_j w_j(g)$ and all continuous seminorms q there holds $q\Big(\sum_{j=0}^{l} a_j x_j\Big) \le \|a\|_H C_q$.

Proof. Clearly (a) implies (b). If we assume (b), then the linear mapping u defined on the space of all $a = \sum_{j=0}^{l} a_j w_j(g) \in p(\mathcal{C})$ by

$$u(a) = \sum_{j=0}^{l} a_j x_j$$

satisfies the inequality $q(u(a)) \le C_q\|a\|_H$ for every $q \in Q$, i.e., u is a continuous linear mapping on $p(\mathcal{C})$ and hence using theorem 1, u admits a unique extension ([10], VI., Prop. 6) $\bar{u}$ that is a continuous linear mapping on $H(\mathcal{C})$ with $\|\bar{u}\|_q \le C_q$ for all $q \in Q$. Since $\bar{u}$ extends u, we obtain $\hat{u}(j) = x_j$. □

We say that the function $F : \mathcal{C} \to X$ is integrable if, for every $x' \in X'$ (the space of all continuous linear forms on X), the function $t \to \big\langle F(g),\, x'\big\rangle$ is Lebesgue integrable, and if, for every $M \in B(\mathcal{C})$ (Borel sets in $\mathcal{C}$), there exists an element $x_M \in X$ such that

$$< x_M, x' >= \int_M \big\langle F(g), x'\big\rangle dg, \quad x' \in X'.$$

If $M = \mathcal{C}$, we write $x_{\mathcal{C}} = \int F(g)dg$ (cf. [9]).

Let (x_j) be a sequence of elements of X. Denote

$$\sigma_N(X, g) = \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) x_j\, w_j(g) \quad N = 1, 2, \ldots$$

and by $S_N(X)$ the continuous linear mapping on $H(\mathcal{C})$ defined by

$$S_N(X)(f) = \frac{1}{2\pi}\int f(g)\sigma_N(X,g)\,dg, \quad f \in H(\mathcal{C}), \quad N = 1,2,\dots$$

If $u \in L(H(\mathcal{C}),X)$, (the linear space of all continuous linear mappings of $H(\mathcal{C})$ into X) and if $x_j = \hat{u}(j)$, we shall write

$$\sigma_N(X,g) = \sigma_N(u,g)\, and\, S_N(X) = S_N(u).$$

We have

$$S_N(X)(f) = \frac{1}{2\pi}\int f(g)\sigma_N(X,g)\,dg =$$

$$\sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big)\hat{f}(j)x_j\,, f \in H(\mathcal{C}).$$

Theorem 4. *The members of a sequence* (x_j) *in* X *are the Walsh-Fourier coefficients of some* $u \in L((H(\mathcal{C})),X)$, *with* $\|u\|_q \le C_q < \infty$, *for all* $q \in Q$, *if and only if* $\left\|S_N(\mathcal{C})\right\|_q \le C_q$, $N = 1,2,\dots$

Proof. The necessity. Let $x_j = \hat{u}(j)$ for some $u \in L(H(\mathcal{C}),X)$ with $\|u\|_q \le C_q$, $q \in Q$. Then $S_N(X) = S_N(u)$, $N = 1,2,\dots$. Recall that $\left\|\sigma_N(f)\right\|_H \le \|f\|_H$ for all $f \in H(\mathcal{C})$. Since, for $f \in H(\mathcal{C})$, we have $S_N(u)(f) = u(\sigma_N(f))$, it follows

$$\begin{aligned}\left\|S_N(X)\right\|_q =& \left\|S_N(u)\right\|_q \\ =& \sup\{q(S_N(u)(f)) : f \in H(\mathcal{C}), \|f\|_H \le 1\} \\ =& \sup\big\{q(u(\sigma_n(f))), f \in H(\mathcal{C}), \|f\|_H \le 1\big\} \\ \le& \sup\big\{q(u(f)) :\ f \in H(\mathcal{C}),\ \|f\|_H \le 1\big\} = \|u\|_q \le C_q,\end{aligned}$$

for all $q \in Q, N = 1,2,\ldots$

The sufficiency. Take $a = \sum_0^l a_j\, w_j(g)$. Then we have

$$\sum_0^l x_j a_j = \lim_{N\to\infty}\sum_0^N\Big(1 - \frac{|j|}{N+1}\Big)x_j a_j = \lim_{N\to\infty} S_N(X)(a)\,.$$

Thus

$$q\bigg(\sum_0^l x_j a_j\bigg) = \lim_{N\to\infty} q(S_N(X)(a)) \le \|a\|_H \limsup\left\|S_N(X)\right\|_q \le \|a\|_H \le C_q\,.$$

According to the theorem 3 there exists $u \in L(H(\mathcal{C}),X)$ such that $x_j = \hat{u}(j)$ and $\|u\|_q \le C_q$ for all $q \in Q$. □

If $F : \mathcal{C} \to X$ is an integrable function, the element of X of the form

$$\frac{1}{2\pi}\int w_j(g)F(g)\,dg\,.$$

is called the Walsh—Fourier—Lebesgue coeficient of F.

Theorem 5. *Let $F : \mathcal{C} \to X$ be an integrable function and put*

$$u(f) = \frac{1}{2\pi}\int f(g)\,F(g)dg, \quad f \in C(\mathcal{C})\,.$$

The members of a sequence (x_j) in X are the Walsh– Fourier—Lebesgue coefficients of F if and only if $\lim_{N\to\infty} S_N(X)(f) = u(f)$ for all $f \in C(\mathcal{C})$.

Proof. Let $x_j = \hat{F}(j)$, $j \in Z$. Clearly $f \to u(f)$ is a continuous linear mapping on $C(\mathcal{C})$ and thus $x_j = \hat{F}(f) = \hat{u}(j)$. By Parseval's formula we have

$$\lim_{N\to\infty} S_N(X)(f) = \lim_{N\to\infty} S_N(u)(f) = \lim_{N\to\infty}\sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big)\hat{f}(j)\hat{u}(j) = u(f)\,,$$

for all $f \in C(\mathcal{C})$. Conversely we have

$$x_j = \lim_{N\to\infty} S_N(X)(w_j(g)) = u(w_j(g))) = \hat{u}(j),$$

i.e.,

$$x_j = \hat{F}(j) = \hat{u}(j) = \frac{1}{2\pi}\int w_j(g)F(g)\,dg.$$

Let now X' be the conjugate of a separable Banach space X. Let $x'(\cdot) : \mathcal{C} \to X'$ be a function such that $x'(\cdot)x$ is measurable for every $x \in X$ and $\operatorname{vrai\,sup}_{t\in\mathcal{C}} \|x'(t)\| = C < \infty$. Then the equality

$$(uf)(x) = \frac{1}{2\pi}\int x'(t)xf(t)\,dt, \quad f \in L^1(\mathcal{C}), \quad x \in X$$

defines a continuous linear mapping $u : L^1(\mathcal{C}) \to X'$ with the norm C([5], VI. 8.6). Hence we may define the jth Walsh-Fourier-Lebesgue coefficient $\hat{x}'(j)$ of such a function $x'(\cdot)$ as the element of X' such that

$$\hat{x}'(j)x = \frac{1}{2\pi}\int x'(t)w_j(t)dt, \quad x \in X.$$

If (x'_j) is a sequence of elements of X', then if we put

$$\sigma_N(X',t) = \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) x'_j\, w_j(t), \quad N = 1, 2, \dots ,$$

for each $x \in X$, the function $\sigma_N(X',\cdot)x$ is measurable and bounded on $\mathcal{C}$. Hence the equation

$$\big(S_N(X')(f)\big)(x) = \frac{1}{2\pi} \int f(t)\sigma_N(X',t)x\, dt, \quad f \in L^1(\mathcal{C})$$

defines a continuous linear mapping $S_N(X')$ of $L^1(\mathcal{C})$ into X' whose norm is $\|S_N(X')\| = \sup_{t\in\mathcal{C}} \|\sigma_N(X',t)\|$ ([5], VI. 8. 6). □

Theorem 6. *Let X' be the conjugate of a separable Banach space X. The members of a sequence (x'_j) of elements of X' are the Walsh—Fourier—Lebesgue coefficients of an essentially unique function $x'(\cdot) : \mathcal{C} \to X'$ such that $x'(\cdot)x$ is measurable and essentially bounded for each $x \in X$, with $\operatorname{vrai\,sup}_{t\in\mathcal{C}} \|x'(t)\| = C < \infty$ if and only if*

$$\|S_N(X')\| \le C, \quad N = 1, 2, \dots$$

Proof. If $x'_j = \hat{x}'(j)$ for some $x'(\cdot) : \mathcal{C} \to X'$ with properties as in the theorem, then, for fixed $t \in \mathcal{C}$ and $x \in X$, we have

$$\begin{aligned}
\big|\sigma_N(X',t)x\big| &= \Big| \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) x'_j x w_j(t) \Big| \\
&= \frac{1}{2\pi} \Big| \int \Big(\sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) w_j(s-t) \Big) x'(s) x ds \Big| \\
&= \frac{1}{2\pi} \Big| \int K_N(s-t) x'(s) x ds \Big| \\
&\le \|K_N\|_1 \big\|x'(\cdot)x\big\|_1 \le C\|x\| ,
\end{aligned}$$

hence $\sup_{g\in\mathcal{C}} \big\|\sigma_N(X',g)\big\| \le C$, $N = 1, 2, \dots$, i.e.,$\big\|S_N(X')\big\| \le C$, $N = 1, 2, \dots$

Conversely, let $\big\|S_N(X')\big\| \le C$, $N = 1, 2, \dots$. Then according to theorem 4 there exists a continuous linear mapping $u : L^1(\mathcal{C}) \to X'$ such that $\hat{u}(j) = x'_j$ and $\|u\| \le C$. Hence there exists ([5], VI. 8.6) an essentially unique function

$x'(\cdot) : \mathcal{C} \to X'$ such that $x'(\cdot)x$ is measurable and essentially bounded for each $x \in X$ and

$$\{u(f)\}(x) = \frac{1}{2\pi} \int x'(g)xf(g)\,dg, \quad f \in L^1(\mathcal{C}), \quad x \in X\,,$$

$$\|u\| = \operatorname*{vrai\,sup}_{s\in\mathcal{C}} \big\|x'(s)\big\| \le C\,.$$

Further, $x'_j = \hat{x}'(j)$. □

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