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ON THE EXISTENCE OF SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS ON UNBOUNDED DOMAINS

Armen L. Beklaryan

Department of Business Analytics
Higher School of Economics
National Research University
33, Kirpichnaya Str., Moscow, 105187, RUSSIA

Abstract: In this paper we consider the first boundary value problem for elliptic systems, defined on unbounded domains $\Omega \subset \mathbb{R}^n$, which solutions satisfy a condition of finiteness of the Dirichlet integral, also known as the energy integral

$$\int\limits_{\Omega} |\nabla u|^2 dx < \infty.$$

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1. Introduction

Let Ω be an unbounded open subset of $\mathbb{R}^n, n \geq 2$. As is customary, by $W^1_{2,\,loc}(\Omega)$ we denote the space of functions which are locally Sobolev, i.e.,

$$W^1_{2,\,loc}(\Omega) = \{f : f \in W^1_2(\Omega \cap B^x_\rho)\,\forall\,\rho > 0\,,\,\forall\,x \in \mathbb{R}^n\},$$

where B^x_ρ the open ball in $\mathbb{R}^n$ of radius ρ centered at the point x [9]. If $x = 0$, we write B_ρ instead of B^x_ρ. In this case, denote by $\overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$ the subset of $W^1_{2,\,loc}(\mathbb{R}^n)$ which is the closure of $C^\infty_0(\Omega)$ in the system of seminorms

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$\|u\|_{W_2^1(\Omega\cap B_\rho)}, \rho > 0$. Further, following [10, Subsec. 1.1], denote by $L_2^1(\Omega)$ the space of distributions ("generalized functions") whose first derivatives belong to $L_2(\Omega)$; in other words,

$$L_2^1(\Omega) = \{f \in \mathcal{D}'(\Omega) : \int\limits_\Omega |\nabla f|^2 dx < \infty\}.$$

Let $\omega \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{K} \subset \omega$ be a compact set. Denote by $\Phi_\varphi(\mathcal{K}, \omega)$ the set of functions $\psi \in C_0^\infty(\omega)$ such that $\psi = \varphi$ in a neighborhood of $\mathcal{K}$, or, in other words, $\psi - \varphi \in \overset{\circ}{W}{}^1_{2,loc}(\mathbb{R}^n \setminus \mathcal{K})$. Write $\Psi(\mathcal{K}, \omega) = \{\psi \in C_0^\infty(\omega) : \psi = 1$ in a neighborhood of $\mathcal{K}$.

The quantity

$$\operatorname{cap}_\varphi(\mathcal{K}, \omega) = \inf_{\psi \in \Phi_\varphi(\mathcal{K}, \omega)} \int\limits_\omega |\nabla \psi|^2 dx$$

is referred to as the capacity of the compact set $\mathcal{K}$ with respect to an open set ω [10, Subsec. 7.2]. The capacity of an arbitrary closed subset $E \subset \omega$ of $\mathbb{R}^n$ is defined by the rule

$$\operatorname{cap}_\varphi(E, \omega) = \sup_{\mathcal{K} \subset E} \operatorname{cap}_\varphi(\mathcal{K}, \omega),$$

where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$. If $\omega = \mathbb{R}^n$, then we write $\operatorname{cap}_\varphi(E)$ instead of $\operatorname{cap}_\varphi(E, \mathbb{R}^n)$.

We also need the following capacity [10, Subsec. 9.1]:

$$\operatorname{Cap}(\mathcal{K}, W_2^1(\omega)) = \inf_{\psi \in \Psi(\mathcal{K}, \omega)} \left(\int\limits_\omega |\nabla \psi|^2 dx + \int\limits_\omega |\psi|^2 \, dx \right).$$

As above, the capacity of an arbitrary set $E \subset \omega$ closed in $\mathbb{R}^n$ is given by the rule

$$\operatorname{Cap}(E, W_2^1(\omega)) = \sup_{\mathcal{K} \subset E} \operatorname{Cap}(\mathcal{K}, W_2^1(\omega)),$$

where the supremum on the right-hand side is taken over all compacta $\mathcal{K} \subset E$.

Finally, denote by W_2^{-1} the space of continuous linear functionals on W_2^1. A set $E \subset \mathbb{R}^n$ is said to be $(2,1)$-*polar* if the only element of W_2^{-1} supported by E is zero [10, Subsec. 9.2].

2. Statement of the Problem

Here and below, L stands for the divergence operator of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

where measurable bounded coefficients a_{ij} satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \le c_2|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad c_1, c_2 > 0.$$

By a solution of the Dirichlet problem

$$\begin{cases} Lu &= 0 \ \text{ on } \Omega \\ u|_{\partial\Omega} &= \varphi, \end{cases} \tag{1}$$

where $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, we mean a function $u \in W^1_{2,\,loc}(\Omega)$ such that

1) $u-\varphi \in \overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$, i.e., $(u-\varphi)\mu \in \overset{\circ}{W}{}^1_2(\Omega)$ for any function $\mu \in C_0^\infty(\mathbb{R}^n)$;
2) the function u has the bounded Dirichlet integral

$$\int\limits_{\Omega} |\nabla u|^2 dx < \infty;$$

3)

$$\int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \psi}{\partial x_i}\,dx = 0$$

for any function $\psi \in C_0^\infty(\Omega)$.

3. Main Results

Theorem 1. *Let cap*$_{\varphi-\,c}(\mathbb{R}^n \setminus \Omega) < \infty$ *for some constant* $c \in \mathbb{R}$. *Then problem* (1) *has a solution.*

Theorem 2. *Let problem* (1) *have a solution, and let*

$$\int\limits_{\mathbb{R}^n\setminus\Omega} |\nabla \varphi|^2 dx < \infty.$$

Then there is a constant $c \in \mathbb{R}$ *such that cap*$_{\varphi-\,c}(\mathbb{R}^n \setminus \Omega) < \infty$.

Theorem 3. *For any function $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, the condition $cap_{\varphi-\,c}(\mathbb{R}^n \setminus \Omega) < \infty$ is equivalent to the inequality*

$$\sum_{k=1}^{\infty} cap_{\varphi-\,c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

where

$$r_k = \begin{cases} 2^k \;, & \text{if } n \geq 3 \\ 2^{2^k} \;, & \text{if } n = 2. \end{cases}$$

Let $\omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let μ be a measure on ω such that

$$\sup_{x \in \mathbb{R}^n,\, \rho > 0} \rho^{1-n} \mu(B^x_\rho \cap \omega) < \infty.$$

In this case, for any function $v \in W^1_2(\omega)$, there is a $c \in \mathbb{R}$ such that

$$\sigma(\omega, \mu) \|v - c\|_{L_2(\omega,\mu)} \leq \|\nabla v\|_{L_2(\omega)}, \tag{2}$$

where the constant $\sigma(\omega, \mu) > 0$ does not depend on v [10, Subsec. 1.4.5].

Theorem 4. *Let problem* (1) *have a solution, and let μ_k be a family of measures on ω_k, where ω_k, $k = 1, 2, \ldots$, are pairwise disjoint Lipschitz domains in $\mathbb{R}^n$ such that*

$$\sup_{x \in \mathbb{R}^n,\, \rho > 0} \rho^{1-n} \mu_k(B^x_\rho \cap \omega_k) < \infty$$

and

$$\sum_{k=1}^{\infty} \int_{\omega_k \setminus \Omega} |\nabla \varphi|^2 dx < \infty. \tag{3}$$

Write

$$m_k(\varphi) = \inf_{c \in \mathbb{R}} \|\varphi - c\|_{L_2(\omega_k \setminus \Omega, \mu_k)}.$$

Then

$$\sum_{k=1}^{\infty} \sigma^2(\omega_k, \mu_k) m^2_k(\varphi) < \infty, \tag{4}$$

where $\sigma(\omega_k, \mu_k)$ stands for the coefficient in inequality (2)*.*

To prove the Theorems 1 – 4 we need a number of auxiliary results.

An inequality from the following lemma is fairly well-known [e.g., 5 p. 288, p. 398] and occurs in various forms. However, for the sake of completeness, we give a detailed proof of this inequality.

Lemma 1 (Special Hardy inequality). *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$. Then*

$$\int_{\mathbb{R}^n} |\nabla\psi|^2\,dx \geq k \int_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\,dx,$$

where constant k doesn't depend on u.

Proof. Let's pass to the polar coordinates. Hence, the integral in the right-hand side takes the form

$$\int dS \int_0^\infty \frac{|\psi|^2}{r^2}\,r^{n-1}\,dr,$$

where first integral is taken over all angular coordinates. Let's fix angular coordinates and obtain a chain of transformations

$$\int_0^\infty \frac{|\psi|^2}{r^2}\,r^{n-1}\,dr = \int_0^\infty |\psi^2|\,r^{n-3}\,dr = \frac{1}{n-2}\int_0^\infty |\psi|^2\,(r^{n-2})'\,dr =$$
$$\frac{1}{n-2}\left(r^{n-2}\,|\psi|^2\Big|_{r=0}^{\infty} - \int_0^\infty 2\,|\psi|\,|\psi|'\,r^{n-2}\,dr \right).$$

The first term in the final bracket, obviously, equals zero, as ψ is a sampling function. Let's estimate the modulus of the second term, using the inequality $ab \leq \varepsilon\,a^2 + \frac{1}{\varepsilon}\,b^2$, considering $a = |\psi|\,r^{\frac{n-3}{2}}$, $b = r^{\frac{n-1}{2}}|\psi'|$.

$$\left|\int_0^\infty 2\,|\psi|\,|\psi|'\,r^{n-2}\,dr\right| \leq 2\int_0^\infty |\psi|\,|\psi'|\,r^{n-2}\,dr \leq$$
$$2\left(\varepsilon\int_0^\infty |\psi|^2\,r^{n-3}\,dr + \frac{1}{\varepsilon}\int_0^\infty |\psi'|^2\,r^{n-1}\,dr\right).$$

Thus, we obtain a chain of inequalities

$$\int_0^\infty |\psi|^2\,r^{n-3}\,dr \leq \frac{2}{n-2}\int_0^\infty |\psi|\,|\psi'|\,r^{n-2}\,dr \leq$$
$$\frac{2\,\varepsilon}{n-2}\int_0^\infty |\psi|^2\,r^{n-3}\,dr + \frac{2}{\varepsilon(n-2)}\int_0^\infty |\psi'|^2\,r^{n-1}\,dr.$$

Consequently, by transferring of the first term to the left-hand side, we obtain following inequality

$$(1-\frac{2\,\varepsilon}{n-2})\int\limits_0^\infty |\psi|^2\, r^{n-3}\, dr \le \frac{2}{\varepsilon(n-2)}\int\limits_0^\infty |\psi'|^2\, r^{n-1}\, dr.$$

Given that $|\psi'|^2 \le |\nabla\psi|^2$ and that r^{n-1} represents the Jacobian of the transformation to the polar coordinates, after returning to the initial coordinates, we obtain

$$\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx \ge k\int\limits_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx.$$

□

Remark. Taking a sequence $\{\psi_k\} \in C_0^\infty(\mathbb{R}^n)$, which is fundamental in L_2^1, i.e. in seminorm

$$\|\cdot\|_{L_2^1(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx\right)^{\frac{1}{2}},$$

by the special Hardy inequality, we immediately obtain a fundamentality of this sequence in the metric

$$\|\cdot\| = \left(\int\limits_{\mathbb{R}^n} |\nabla\psi|^2\, dx + \int\limits_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2}\, dx\right)^{\frac{1}{2}}.$$

Therefore, the special Hardy inequality is also fair for $\psi \in \overset{\circ}{L}{}_2^1(\mathbb{R}^n)$.

Lemma 2 (General Hardy inequality). *Let $u \in L_2^1(\mathbb{R}^n)$ and $n \ge 3$. Then there is a constant c such that the following inequality is fair*

$$\int\limits_{\mathbb{R}^n} |\nabla u|^2\, dx \ge k\int\limits_{\mathbb{R}^n} \frac{|u-c|^2}{|x|^2}\, dx,$$

where constant k doesn't depend on u.

Proof. The fact that u belongs to the space $L_2^1(\mathbb{R}^n)$ is equivalent to the following condition

$$\int\limits_{\mathbb{R}^n} |\nabla u|^2\, dx < \infty.$$

Let's decompose the space $L_2^1(\mathbb{R}^n)$ into a direct product of the space $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ and its orthogonal complement. Let u_0 be a projection of u on $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$, and h is a component from the orthogonal complement. Considering that the space $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ is Hilbert and separable, we find out that for any $v \in \overset{\circ}{L}{}_2^1(\mathbb{R}^n)$ it is true that

$$\int\limits_{\mathbb{R}^n} \nabla v \nabla h \, dx = 0.$$

Hence, $\triangle h = 0$ in $\mathbb{R}^n$. From the Parseval's identity, we obtain that

$$\int\limits_{\mathbb{R}^n} |\nabla h|^2 \, dx + \int\limits_{\mathbb{R}^n} |\nabla u_0|^2 \, dx = \int\limits_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Due to the finiteness of the right-hand side, we have the finiteness of each term on the left-hand side. In particular, we obtain that

$$\int\limits_{\mathbb{R}^n} |\nabla h|^2 \, dx < \infty.$$

Recalling the ellipticity of h, we obtain that h is constant. Then, using the special Hardy inequality with respect to $u_0 = u - h = u - c$, we obtain the general Hardy inequality. □

Lemma 3. *In case of $n = 2$, the general Hardy inequality takes the form*

$$\int\limits_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq k \int\limits_{|x| \geq 2\delta} \frac{|u|^2}{|x|^2 \ln^2 \frac{|x|}{\delta}} \, dx,$$

for any function $u \in L_2^1(\mathbb{R}^2)$ and for any constant $\delta > 0$, where constant k doesn't depend on u, which is equivalent to the inequality

$$\int\limits_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq k \int\limits_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \ln^2 |x|} \, dx,$$

for any function $u \in L_2^1(\mathbb{R}^2)$ such that $u = 0$ almost everywhere in a neighborhood of zero, and where constant k doesn't depend on u.

Proof. At first let's prove this proposition for a function $u \in C^\infty(\mathbb{R}^2)$.

Let's pass to the polar coordinates. Hence, the integral in the right-hand side takes the form

$$\int_0^{2\pi} d\phi \int_0^{\infty} \frac{r|u|^2}{r^2 \ln^2 r}\, dr.$$

Let's fix angular coordinates and obtain a chain of transformations

$$\int_0^{\infty} \frac{r|u|^2}{r^2 \ln^2 r}\, dr = \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr = -\int_0^{\infty} \left(\frac{1}{\ln r}\right)' |u|^2\, dr =$$

$$-\frac{1}{\ln r}\, |u|^2 \Big|_{r=0}^{\infty} + \int_0^{\infty} \frac{1}{\ln r}\, 2\, |u||u|' dr.$$

The first term in the final bracket, obviously, is zero, as u vanishes in a neighborhood of zero. Let's estimate the modulus of the second term, using the inequality $ab \le \varepsilon\, a^2 + \frac{1}{\varepsilon}\, b^2$, considering $a = \frac{|u|}{r^{\frac{1}{2}} \ln r}$, $b = r^{\frac{1}{2}}|u'|$.

$$\left|\int_0^{\infty} \frac{1}{\ln r} 2|u||u|' dr\right| = \left|\int_0^{\infty} \frac{2r^{\frac{1}{2}}|u||u|'}{r^{\frac{1}{2}} \ln r}\, dr\right| \le 2\int_0^{\infty} \frac{r^{\frac{1}{2}}|u||u'|}{r^{\frac{1}{2}} \ln r} dr \le$$

$$2\left(\varepsilon \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr + \frac{1}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr\right).$$

Thus, we obtain a chain of inequalities

$$\int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr \le 2\,\varepsilon \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr + \frac{2}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr.$$

Consequently, by transferring of the first term to the left-hand side, we obtain the following inequality

$$(1 - 2\,\varepsilon) \int_0^{\infty} \frac{|u|^2}{r \ln^2 r}\, dr \le \frac{2}{\varepsilon} \int_0^{\infty} r|u'|^2\, dr.$$

Given that $|u'|^2 \le |\nabla u|^2$ and that r represents the Jacobian of the transformation to the polar coordinates, after returning to the initial coordinates, we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx \ge k \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \ln^2 |x|}\, dx.$$

Then, using the passage to the limit and the fact that C^∞ is dense in L_2^1 [10, p. 18], we obtain the proof of the Lemma. $\square$

Lemma 4. *Let E be a $(2,1)$-polar set. Then $u\big|_E = 0$ for any function $u \in W^1_{2,\,loc}(\mathbb{R}^n)$, i.e. $\mu u \in \overset{\circ}{W}{}^1_2(\mathbb{R}^n \setminus E)$ for any function $\mu \in C_0^\infty(\mathbb{R}^n)$.*

Proof. It is known [10, p. 331, Theorem 1] that the space $\mathcal{D}(\Omega)$ is dense in W_2^1 if and only if $\mathbb{R}^n \setminus \Omega$ is a $(2,1)$-polar set. That implies the statement of the Lemma. $\square$

Lemma 5. *Let $Cap((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) > 0$ for some r_0. Then*

$$\|\varphi\|_{L_2(B_r)} \le A\,\|\nabla\varphi\|_{L_2(B_r)}$$

for any $r > 2\,r_0$ and for any $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$ such that

$$\varphi\Big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_{r_0}} = 0,$$

where constant A doesn't depend on φ.

Proof. Let's suppose the contrary. Then for any constant A there is $r > 2\,r_0$ and a function $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$ such that

$$\varphi\Big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_{r_0}} = 0,$$

and besides it is true that

$$\|\varphi\|_{L_2(B_r)} > A\,\|\nabla\varphi\|_{L_2(B_r)}.$$

Let's choose a sequence $A_s = s,\ s = 1, 2, \dots$ There is a sequence φ_s such that $\|\varphi_s\|_{L_2(B_r)} > s\,\|\nabla\varphi_s\|_{L_2(B_r)}$. Denote

$$\psi_s = \frac{\varphi_s}{\|\varphi_s\|_{L_2(B_r)}}.$$

It is obvious that $\|\psi_s\|_{L_2(B_r)} = 1$, while

$$\|\nabla\psi_s\|_{L_2(B_r)} \to 0 \ \text{ as } s \to \infty.$$

Consequently, $\|k-\psi_s\|_{W_2^1(B_r)}$ tends to zero as $s \to \infty$ for some constant k. Thus, taking the function $(k-\psi_s)\eta$, where $\eta \in C_0^\infty(B_{2r_0})$, $\eta \equiv 1$ in a neighborhood of B_{r_0}, we obtain that

$$\operatorname{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) \leq \int\limits_{\mathbb{R}^n} |\nabla((k-\psi_s)\eta)|^2 dx \leq \text{const}\, \|k-\psi_s\|_{W_2^1(B_r)}.$$

Taking the limit as $s \to \infty$, it follows that $\operatorname{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_0}, W_2^1(\mathbb{R}^n)) = 0$. This contradiction proves the Lemma. □

Proof of the Theorem 1. $\{r_i\}_{i=1}^n$ and $\{\rho_i\}_{i=1}^n$ are infinitely increasing sequences of real numbers. Let $r_i < \rho_i$ for all i, and

$$\operatorname{cap}_{\varphi-c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, B_{\rho_i}) < \operatorname{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i}, \ i = 1, 2, \ldots$$

It is obvious [11] that capacity $\operatorname{cap}_{\varphi-c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, B_{\rho_i})$ is achieved by the function $v_i \in \overset{\circ}{W}{}_2^1(B_{\rho_i})$ such that

$$\begin{cases} \triangle v_i = 0 \quad B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \\ v_i \Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}} = \varphi - c, \end{cases} \tag{5}$$

where the last equality means that $(v_i - (\varphi - c))\mu \in \overset{\circ}{W}{}_2^1(B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}))$ for any $\mu \in C_0^\infty(B_{\rho_i})$. Along with the problem (5), let's consider another problem:

$$\begin{cases} Lu_i = 0 \ \text{ on } B_{\rho_i} \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \\ u_i \Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}} = \varphi - c, \end{cases} \tag{6}$$

where $u_i \in \overset{\circ}{W}{}_2^1(B_{\rho_i})$.

The following statement takes place: let function u_i be a solution of the problem (6), and function v_i is a solution of the problem (5). Then

$$\int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx \leq \int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \leq c \int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx\,, \tag{7}$$

where c is a non-negative constant, which doesn't depend on u_i and v_i. Let's prove this fact. The left-hand inequality, obviously, follows from the definition of capacity. Let's prove the right-hand inequality. Given that the function u_i is a solution of the problem (6), it is true that

$$\int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial \psi}{\partial x_l}\, dx = 0$$

for any function $\psi \in \overset{\circ}{W}{}_2^1(\Omega)$. In particular, taking $\psi = u - v$, we obtain

$$\int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_l}\, dx - \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial v_i}{\partial x_l}\, dx = 0,$$

from where the following estimates are obtained

$$\gamma \int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \leq \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial u_i}{\partial x_l}\, dx = \int\limits_{B_{\rho_i}} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial u_i}{\partial x_m} \frac{\partial v_i}{\partial x_l}\, dx \leq$$

$$const \left(\int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\nabla v_i|^2 dx \right)^{\frac{1}{2}},$$

that prove the right-hand inequality in (7).

It is obvious that

$$\mathrm{cap}_{\,\varphi - c}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}) \leq \int\limits_{B_{\rho_i}} |\nabla v_i|^2 dx < \mathrm{cap}_{\,\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i}.$$

At the same time, from the inequality (7), it follows that

$$\int\limits_{B_{\rho_i}} |\nabla u_i|^2 dx < c \left(\mathrm{cap}_{\,\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i} \right).$$

If $\mathrm{Cap}(\mathbb{R}^n \setminus \Omega, W_2^1(\mathbb{R}^n)) = 0$ then the set $\mathbb{R}^n \setminus \Omega$ is $(2,1)$-polar [10, p. 331], what means, according to the Lemma 4, that the function $1 - \varphi$ is zero on $\mathbb{R}^n \setminus \Omega$. Thus, taking the unit function, we obtain the required solution of the problem (1).

Now let $\mathrm{Cap}((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{r_i}, W_2^1(\mathbb{R}^n)) > 0$ for some r_i. Then, from the Lemma 5, we obtain that the sequence $\{u_i\}_{i=1}^n$ is bounded both in $L_2(B_r)$ and $W_2^1(B_r)$ for any r. Indeed, for sufficiently large i, j it is true that

$$u_i - u_j \in \overset{\circ}{W}{}^1_{2,\,loc}(\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r)), \quad i, j > i_0.$$

Thus, fixing j, we have

$$\|u_i\|^2_{L^1_2} + \|u_i\|^2_{W^1_2(K)} \le \alpha \left(\|u_j\|^2_{W^1_2(K)} + cap_{\varphi - c}(\mathbb{R}^n \setminus \Omega) + \frac{1}{2^i} + \frac{1}{2^j} \right)$$

for any compacta $K \subset \mathbb{R}^n$, where constant $\alpha > 0$ doesn't depend on u_i.

Due to the compactness of the embedding $W^1_2(B_r)$ in $L_2(B_r)$, we can choose a subsequence of the sequence $\{u_i\}_{i=1}^n$ which is fundamental in $L_2(B_r)$. In order not to overload the indexes, we denote this subsequence also as $\{u_i\}_{i=1}^n$. Let's take a function $\eta \in C_0^\infty(B_r)$ such that $\eta \equiv 1$ in an open neighborhood of the set $\overline{B}_{r/2}$. Due to the fact that u_i satisfies (6), for the difference $u_i - u_j$ we obtain that

$$\int\limits_{B_r} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial (u_i - u_j)}{\partial x_m} \frac{\partial \psi}{\partial x_l}\, dx = 0,$$

where $\psi = \eta^2\,(u_i - u_j)$. In other words,

$$\int\limits_{B_r} \sum_{l,m=1}^{n} a_{lm}(x) \frac{\partial (u_i - u_j)}{\partial x_m} \frac{\partial \eta^2}{\partial x_l} (u_i - u_j)\, dx + \int\limits_{B_r} \sum_{l,m=1}^{n} a_{lm}(x) \eta^2 \,\frac{\partial (u_i - u_j)}{\partial x_m}\, \frac{\partial (u_i - u_j)}{\partial x_l}\, dx = 0.$$

Let's rewrite the last relation in the form

$$\gamma \int\limits_{B_r} \eta^2 |\nabla (u_i - u_j)|^2 dx \le -2 \int\limits_{B_r} \sum_{k,l=1}^{n} a_{kl}(x) \frac{\partial (u_i - u_j)}{\partial x_l} \frac{\partial \eta}{\partial x_k}\, \eta\, (u_i - u_j)\, dx,$$

whence, in view of the inequality $ab \le \frac{1}{2}\,a^2 + \frac{1}{2}\,b^2$, we obtain that

$$\int\limits_{B_r} \eta^2 |\nabla (u_i - u_j)|^2 dx \le c_1 \int\limits_{B_r} |\nabla (u_i - u_j)|^2 \eta^2 dx + c_2 \int\limits_{B_r} |\nabla \eta|^2 (u_i - u_j)^2\, dx,$$

where c_1, c_2 are non-negative constants, which don't depend on u_i. Thus, we have

$$\int\limits_{B_{r/2}} |\nabla (u_i - u_j)|^2 dx \le \beta \int\limits_{B_r} (u_i - u_j)^2\, dx,$$

where constant $\beta > 0$ doesn't depend on u_i. Last inequality proves that the sequence $\{u_i\}_{i=1}^n$ is fundamental in $W^1_2(B_{r/2})$ for any $r > 0$. Therefore, there is a function $u \in W^1_{2,loc}(\mathbb{R}^n)$ such that for any $r > 0$ the sequence $\{u_i\}_{i=1}^n$ tends to u in $W_2^1(B_r)$. It is obvious that the function u is the desired solution of the problem (1). □

Proof of the Theorem 2. Let's suppose that the function u is a solution of the problem (1). Let's extend u on $\mathbb{R}^n \setminus \Omega$ with value φ. Let $n \geq 3$, then there is a constant $c \in \mathbb{R}$ such that for the function u the general Hardy inequality takes place. Denote

$$\nu_R = \eta\left(\frac{|x|}{R}\right)(u-c),$$

where $\eta \in C_0^\infty(B_2)$ and $\eta \equiv 1$ in an open neighborhood of the set $\overline{B}_1$. Hence, we obtain

$$\nu_R\big|_{(\mathbb{R}^n\setminus\Omega)\cap\overline{B}_R} = \varphi - c.$$

The Dirichlet integral for the function ν_R can be estimated

$$\int\limits_{B_{2R}} \left|\nabla\left(\eta\left(\frac{|x|}{R}\right)(u-c)\right)\right|^2 dx \leq$$

$$2\left(\int\limits_{B_{2R}} \left|\nabla\eta\left(\frac{|x|}{R}\right)(u-c)\right|^2 dx + \int\limits_{B_{2R}} \left|\eta\left(\frac{|x|}{R}\right)\nabla(u-c)\right|^2 dx\right).$$

Let's notice that

$$\left|\nabla\eta\left(\frac{|x|}{R}\right)\right| \leq \frac{p}{R} \text{ and } \frac{1}{R^2} \leq \frac{4}{|x|^2} \text{ for } x \in B_{2R},$$

where $p > 0$ is a constant. Then, considering the Hardy inequality, we obtain

$$\int\limits_{B_{2R}} \left|\nabla\eta\left(\frac{|x|}{R}\right)(u-c)\right|^2 dx \leq \frac{p^2}{R^2}\int\limits_{B_{2R}\setminus B_R} |u-c|^2\, dx \leq$$

$$4p^2 \int\limits_{B_{2R}\setminus B_R} \frac{|u-c|^2}{|x|^2}dx \leq \frac{4p^2}{k}\int\limits_{B_{2R}} |\nabla u|^2 dx.$$

Thus,

$$\operatorname{cap}_{\varphi-c}((\mathbb{R}^n\setminus\Omega)\cap\overline{B}_R) \leq \int\limits_{B_{2R}} |\nabla\nu_R|^2 dx \leq \gamma\int\limits_{B_{2R}} |\nabla u|^2 dx < \infty,$$

where $\gamma > 0$ is a constant, which doesn't depend on ν_R. Proceeding to the limit as $R \to \infty$, we obtain

$$\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) \leq \gamma \int\limits_{\mathbb{R}^n} |\nabla u|^2 dx < \infty,$$

that proves the Theorem 2 for $n \geq 3$.

In case of $n = 2$, denoting

$$\nu_R = \eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) u,$$

where $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta = 0$ in a neighborhood of zero and $\eta \equiv 1$ in a open neighborhood of B_1, we obtain

$$\nu_R \Big|_{(\mathbb{R}^n \setminus \Omega) \cap \overline{B}_{R^2}} = \varphi.$$

The Dirichlet integral for the function ν_R can be estimated

$$\int\limits_{B_{2R^2}} \left| \nabla \left(\eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) u \right) \right|^2 dx \leq$$

$$2 \left(\int\limits_{B_{2R^2}} \left| \nabla \eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) u \right|^2 dx + \int\limits_{B_{2R^2}} \left| \eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) \nabla u \right|^2 dx \right).$$

Let's notice that

$$\left| \nabla \eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) \right| \leq \frac{2q}{|x| \ln R^2} \text{ and } \frac{1}{\ln^2 R^2} \leq \frac{m}{\ln^2 |x|} \text{ for } x \in B_{2R^2},$$

where $q, m > 0$ are some constants. Then, considering the Hardy inequality, we obtain

$$\int\limits_{B_{2R^2}} \left| \nabla \eta \left(\frac{\ln \frac{|x|}{R}}{\ln R} \right) u \right|^2 dx \leq \int\limits_{B_{2R^2} \setminus B_{R^2}} \frac{4\, q^2}{|x|^2 \ln^2 R^2} |u|^2 \, dx \leq$$

$$4\, q^2 m \int\limits_{B_{2R^2} \setminus B_{R^2}} \frac{|u|^2}{|x|^2 \ln^2 |x|} dx \leq \frac{4 q^2 m}{k} \int\limits_{B_{2R^2}} |\nabla u|^2 dx.$$

Thus,

$$\operatorname{cap}_{\varphi}((\mathbb{R}^2 \setminus \Omega) \cap \overline{B}_{R^2}) \leq \int\limits_{B_{2R^2}} |\nabla \nu_R|^2 dx \leq \gamma \int\limits_{B_{2R^2}} |\nabla u|^2 dx < \infty,$$

where $\gamma > 0$ is a constant, which doesn't depend on ν_R. Proceeding to the limit as $R \to \infty$, we obtain

$$\operatorname{cap}_{\varphi}(\mathbb{R}^2 \setminus \Omega) \leq \gamma \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx < \infty.$$

Thus, the Theorem 2 is completely proved. □

Proof of the Theorem 3. Let $\operatorname{cap}_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$. Then, by the Theorem 2, there is a function u, which is a solution of the problem (1). Let $n \geq 3$. Let's consider a shearing function $\eta_k \in C_0^\infty(\mathbb{R}^n)$ such that $\eta_k(x) = 1$ on $B_{2^{k+1}} \setminus B_{2^{k-1}}$ and $\operatorname{supp} \eta_k(x) \subset B_{2^{k+2}} \setminus B_{2^{k-2}}$, $k = 1, 2, \ldots$, which is constructed as follows. Let $\eta(x)$ is a monotone non-decreasing function from $C^\infty(\mathbb{R}^n)$, which is equal to zero on the interval $[-\infty, \frac{1}{4}]$ and is equal to one on the interval $[\frac{3}{4}, +\infty]$. Further, we denote by $\eta_k(x)$ the following function

$$\eta_k(x) = \begin{cases} \eta\left(\dfrac{|x| - r_{k-2}}{r_{k-1} - r_{k-2}}\right), & \text{if } x \in \overline{B}_{r_{k-1}} \setminus B_{r_{k-2}} \\ 1, & \text{if } x \in \overline{B}_{r_{k+1}} \setminus B_{r_{k-1}} \\ \eta\left(\dfrac{r_{k+2} - |x|}{r_{k+2} - r_{k+1}}\right), & \text{if } x \in \overline{B}_{r_{k+2}} \setminus B_{r_{k+1}}. \end{cases}$$

We have the estimate

$$|\nabla \eta_k(x)|^2 \leq \frac{c}{|x|^2},$$

where c doesn't depend on k. Then, considering the Hardy inequality, we obtain a chain of inequalities

$$\operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$

$$\int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla (\eta_k(x) u(x))|^2 \, dx \leq 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla \eta_k(x) u(x)|^2 \, dx +$$

$$2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\eta_k(x) \nabla u(x)|^2 \, dx \leq 2c \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx +$$

$$b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx,$$

where b_1 is a positive constant, which doesn't depend on u. Thus,

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$
$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx.$$

Due to the fact that each point $x \in \mathbb{R}^n$ belongs to no more than three areas $B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}$, we obtain

$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2} \, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}} |\nabla u(x)|^2 \, dx \leq$$
$$b_2 \int\limits_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx + b_3 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \leq b_4 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx < \infty,$$

where b_2, b_3, b_4 are positive constants, which don't depend on u.

Now let $n = 2$. Denote

$$\eta_k(x) = \begin{cases} \eta\left(\dfrac{\ln \frac{|x|}{r_{k-2}}}{\ln \frac{r_{k-1}}{r_{k-2}}}\right), & \text{if } x \in \overline{B}_{r_{k-1}} \setminus B_{r_{k-2}} \\ 1\,, & \text{if } x \in \overline{B}_{r_{k+1}} \setminus B_{r_{k-1}} \\ \eta\left(\dfrac{\ln \frac{r_{k+2}}{|x|}}{\ln \frac{r_{k+2}}{r_{k+1}}}\right), & \text{if } x \in \overline{B}_{r_{k+2}} \setminus B_{r_{k+1}}, \end{cases}$$

where $\eta(x)$ is a monotone non-decreasing function from $C^\infty(\mathbb{R}^n)$, which is equal to zero on the interval $[-\infty, \frac{1}{4}]$ and is equal to one on the interval $[\frac{3}{4}, +\infty]$. We have the estimate

$$|\nabla \eta_k(x)|^2 \leq \frac{c}{|x|^2 \ln^2 |x|},$$

where c doesn't depend on k. Then, considering the Hardy inequality, we obtain a chain of inequalities

$$\operatorname{cap}_{\varphi - c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \leq$$

$$\int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\nabla(\eta_k(x)u(x))|^2\, dx \le 2 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\nabla\eta_k(x)u(x)|^2\, dx +$$

$$2 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\eta_k(x)\nabla u(x)|^2\, dx \le 2c \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\, dx +$$

$$b_1 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\, dx,$$

where b_1 is a positive constant, which doesn't depend on u. Thus,

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) \le$$

$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\, dx.$$

Due to the fact that each point $x \in \mathbb{R}^n$ belongs to no more than three areas $B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}$, we obtain

$$\sum_{k=1}^{\infty} 2 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\, dx + \sum_{k=1}^{\infty} b_1 \int\limits_{B_{r_{k+2}}\backslash\overline{B}_{r_{k-2}}} |\nabla u(x)|^2\, dx \le$$

$$b_2 \int\limits_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2\ln^2|x|}\, dx + b_3 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2\, dx \le b_4 \int\limits_{\mathbb{R}^n} |\nabla u(x)|^2\, dx < \infty,$$

where b_2, b_3, b_4 are positive constants, which don't depend on u.

The converse. Let

$$\sum_{k=1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty$$

and $n \ge 3$. Let's consider a shearing function

$$\widetilde{\psi}_k(x) = \begin{cases} \eta\left(\dfrac{|x| - r_{k-1}}{r_k - r_{k-1}}\right), & \text{if } |x| \le r_k \\ \eta\left(\dfrac{r_{k+1} - |x|}{r_{k+1} - r_k}\right), & \text{if } |x| \ge r_k. \end{cases}$$

Denote

$$\psi_k(x) = \frac{\widetilde{\psi}_k(x)}{\sum\limits_{i=0}^{\infty} \widetilde{\psi}_i(x)}.$$

Obviously,

$$\sum_{k=1}^{\infty} \psi_k(x) = 1.$$

From the condition on the capacity, we have functions $u_k(x)$, which implement the capacity and equal to $\varphi - c$ on $B_{r_{k+1}} \setminus B_{r_{k-1}}$ and with supports from $B_{r_{k+2}} \setminus B_{r_{k-2}}$. Let's notice that

$$\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) = \varphi - c,$$

if x is from a neighborhood of the set $(\overline{B}_{r_{N_2-1}} \setminus B_{r_{N_1+1}}) \cap (\mathbb{R}^n \setminus \Omega)$. Then we obtain

$$\left|\nabla \sum_{k=N_1}^{N_2} u_k(x)\psi_k(x)\right|^2 = \left|\sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x) + \sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x)\right|^2 \leq$$

$$2\left|\sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x)\right|^2 + 2\left|\sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x)\right|^2.$$

Since for each $x \in \mathbb{R}^n$ there are no more than three natural numbers $k \in \{N_1, \ldots, N_2\}$ such that $\psi_k(x) \neq 0$, then we obtain

$$\left|\sum_{k=N_1}^{N_2} \nabla u_k(x)\psi_k(x)\right|^2 \leq 9 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2.$$

Similarly,

$$\left|\sum_{k=N_1}^{N_2} u_k(x)\nabla\psi_k(x)\right|^2 \leq 9 \sum_{k=N_1}^{N_2} |u_k(x)|^2 |\nabla\psi_k(x)|^2.$$

As a result, we obtain

$$\left|\nabla \sum_{k=N_1}^{N_2} u_k(x)\psi_k(x)\right|^2 \leq 18 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2 + 18 \sum_{k=N_1}^{N_2} |\nabla u_k(x)|^2 |\psi_k(x)|^2.$$

Therefore,

$$\int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right) \right|^2 dx \le 18 \left(\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx + \right.$$

$$\left. \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla \psi_k(x)|^2 dx \right).$$

The first term in the last expression can be estimated as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

since $|\psi_k(x)| \le 1$. Using the fact that

$$|\nabla \psi_k(x)| \le \frac{r_{k+1} - r_k}{2^k}$$

and Friedrichs' inequality, we estimate the second term as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)\nabla \psi_k(x)|^2 dx \le \sum_{k=N_1}^{N_2} \frac{(r_{k+1} - r_k)^2}{4^k} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x)|^2 dx \le$$

$$c_1 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

where c_1 is a positive constant, which doesn't depend on u_k and ψ_k. We obtain a chain of inequalities

$$\operatorname{cap}_{\varphi-c}((\overline{B}_{r_{N_2}} \setminus B_{r_{N_1}}) \cap (\mathbb{R}^n \setminus \Omega)) \le \int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) \right) \right|^2 dx \le$$

$$c_2 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx =$$

$$\sum_{k=N_1}^{N_2} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}),$$

where c_2 is a positive constant, which doesn't depend on u_k and ψ_k. As N_2 tending to infinity, we obtain

$$\operatorname{cap}_{\varphi-c}((\mathbb{R}^n \setminus \Omega) \setminus B_{r_{N_1}}) \leq$$
$$\sum_{k=N_1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

what implies that

$$\operatorname{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty.$$

Now let $n = 2$. Then let's consider a shearing function

$$\widetilde{\psi}_k(x) = \begin{cases} \eta\left(\dfrac{\ln \frac{|x|}{r_{k-1}}}{\ln \frac{r_k}{r_{k-1}}}\right), & \text{if } |x| \leq r_k \\ \eta\left(\dfrac{\ln \frac{r_{k+1}}{|x|}}{\ln \frac{r_{k+1}}{r_k}}\right), & \text{if } |x| \geq r_k, \end{cases}$$

and let

$$\psi_k(x) = \frac{\widetilde{\psi}_k(x)}{\sum\limits_{i=0}^{\infty} \widetilde{\psi}_i(x)}.$$

Obviously,

$$\sum_{k=1}^{\infty} \psi_k(x) = 1.$$

From the condition on the capacity, we have functions $u_k(x)$, which implement the capacity and equal to $\varphi - c$ on $B_{r_{k+1}} \setminus B_{r_{k-1}}$ and with supports from $B_{r_{k+2}} \setminus B_{r_{k-2}}$. Let's notice that

$$\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x) = \varphi - c,$$

if x is from a neighborhood of the set $(\overline{B}_{r_{N_2-1}} \setminus B_{r_{N_1+1}}) \cap (\mathbb{R}^n \setminus \Omega)$. It is easy to see that the functions ψ_k again satisfy the following relations

$$\int_{\mathbb{R}^n} \left|\nabla\left(\sum_{k=N_1}^{N_2} u_k(x)\psi_k(x)\right)\right|^2 dx \leq 18\left(\sum_{k=N_1}^{N_2} \int_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)\psi_k(x)|^2 dx + \right.$$

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x) \nabla \psi_k(x)|^2 dx \Bigg) .$$

The first term in the last expression can be estimated as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x) \psi_k(x)|^2 dx \leq \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

since $|\psi_k(x)| \leq 1$. Then, using the fact that $|\nabla \psi_k(x)| \leq \dfrac{\text{const}}{|x| \ln |x|}$ and the Hardy inequality, we estimate the second term as follows

$$\sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |u_k(x) \nabla \psi_k(x)|^2 dx \leq \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} \frac{|u_k(x)|^2}{|x|^2 \ln^2 |x|} dx \leq$$

$$c_1 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx,$$

where c_1 is a positive constant, which doesn't depend on u_k and ψ_k. We obtain a chain of inequalities

$$\operatorname{cap}_{\varphi-c}((\overline{B}_{r_{N_2}} \setminus B_{r_{N_1}}) \cap (\mathbb{R}^n \setminus \Omega)) \leq \int\limits_{\mathbb{R}^n} \left| \nabla \left(\sum_{k=N_1}^{N_2} u_k(x) \psi_k(x) \right) \right|^2 dx \leq$$

$$c_2 \sum_{k=N_1}^{N_2} \int\limits_{B_{r_{k+2}} \setminus B_{r_{k-2}}} |\nabla u_k(x)|^2 dx =$$

$$\sum_{k=N_1}^{N_2} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}),$$

where c_2 is a positive constant, which doesn't depend on u_k and ψ_k. As N_2 tending to infinity, we obtain

$$\operatorname{cap}_{\varphi-c}((\mathbb{R}^n \setminus \Omega) \setminus B_{r_{N_1}}) \leq$$

$$\sum_{k=N_1}^{\infty} \operatorname{cap}_{\varphi-c}((\overline{B}_{r_{k+1}} \setminus B_{r_{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{r_{k+2}} \setminus \overline{B}_{r_{k-2}}) < \infty,$$

what implies that
$$\operatorname{cap}_{\varphi-c}(\mathbb{R}^n \setminus \Omega) < \infty.$$

□

Proof of the Theorem 4. Let u is a solution of the problem (1). Let's extend u on $\mathbb{R}^n \setminus \Omega$ with value φ. Then, by the inequality (2), we obtain that
$$\sigma(\omega_k, \mu_k)\|u - c_k\|_{L_2(\omega_k,\mu_k)} \le \|\nabla u\|_{L_2(\omega_k)},$$
what implies that
$$\sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} = \sigma^2(\omega_k, \mu_k)\|u - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} \le \|\nabla u\|^2_{L_2(\omega_k)}.$$
Summing this relation, we obtain
$$\sum_{k=1}^{\infty}\sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} \le \sum_{k=1}^{\infty}\|\nabla u\|^2_{L_2(\omega_k)} =$$
$$\sum_{k=1}^{\infty}\int\limits_{\omega_k\cap\Omega}|\nabla u|^2dx + \sum_{k=1}^{\infty}\int\limits_{\omega_k\setminus\Omega}|\nabla u|^2dx.$$
Let's notice that
$$\sum_{k=1}^{\infty}\int\limits_{\omega_k\cap\Omega}|\nabla u|^2dx \le \int\limits_{\Omega}|\nabla u|^2dx < \infty,$$
and
$$\sum_{k=1}^{\infty}\int\limits_{\omega_k\setminus\Omega}|\nabla u|^2dx = \sum_{k=1}^{\infty}\int\limits_{\omega_k\setminus\Omega}|\nabla \varphi|^2dx < \infty.$$
Thus, we have
$$\sum_{k=1}^{\infty}\sigma^2(\omega_k, \mu_k)\|\varphi - c_k\|^2_{L_2(\omega_k\setminus\Omega,\mu_k)} < \infty,$$
which immediately implies (4). The Theorem is completely proved. □

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