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A PARAMETER FOR RAMANUJAN'S FUNCTION $\chi(q)$ OF DEGREE 9 AND THEIR EXPLICIT EVALUATION

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Abstract: In this paper, we study a ratio's of parameter for Ramanujan's function $\chi(q)$ of degree 9 and their explicit values.

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1. Introduction

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$\begin{aligned} f(a,b) :=& \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |\, ab \,| < 1, \\ =&(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \end{aligned} \tag{1}$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

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Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{4}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{5}$$

Now we define a modular equation in brief. The ordinary hypergeometric series
${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $\mid x \mid < 1$.
Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right) \tag{6}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right), \tag{7}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}. \tag{8}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (8). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

The function $\chi(q)$ is intimately connected to Ramanujan's class invariants G_n an g_n which are defined by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q), \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q) \tag{9}$$

where $q = e^{-\pi\sqrt{n}}$ and n is a positive rational number. Since from [1, Entry 12(v),(vi), p.56]

$$\chi(q) = 2^{1/6}\left\{\alpha(1-\alpha)q^{-1}\right\}^{-1/24} \tag{10}$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}\alpha^{-1/24}q^{-1/24} \tag{11}$$

Nipen Saikia [4] introduce the parameter $I_{m,n}$ which is defined as

$$I_{m,n} := \frac{\chi(q)}{q^{(-m+1)/24}\chi(q^m)}, \quad q = e^{-\pi\sqrt{n/m}}, \tag{12}$$

where m and n are positive real numbers.

In Section 3, we study the modular relation between $I_{9,n}$ and I_{9,k^2n}, their explicit evaluations of $I_{9,n}$ for $n = 2, 3, 5$ and 7.

2. Preliminary Results

Lemma 1. *[3] If* $P = \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^9)\varphi(q^{18})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{18})}{\varphi(q^9)\varphi(q^2)}$, *then*

$$\begin{aligned} Q^2 + \frac{1}{Q^2} + \left(P^2 + \frac{9^2}{P^2}\right) + 2\left(P + \frac{9}{P}\right)\left[4 + 3\left(Q + \frac{1}{Q}\right)\right] = 8\left(Q + \frac{1}{Q}\right) \\ + 4\left[\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - \left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 3\right]. \end{aligned} \tag{13}$$

Lemma 2. *[3] If* $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{27})}$, *then*

$$\left(3 - P - \frac{3}{P}\right)\left(3 - Q - \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2. \tag{14}$$

Lemma 3. *[3] If* $P = \dfrac{\varphi(q)\varphi(q^5)}{\varphi(q^9)\varphi(q^{45})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{45})}{\varphi(q^9)\varphi(q^5)}$*, then*

$$Q^3 + \frac{1}{Q^3} - 15\left(Q^2 + \frac{1}{Q^2}\right) - 45\left(Q + \frac{1}{Q}\right) - \left(P^2 + \frac{81}{P^2}\right) - 10\left(P + \frac{9}{P}\right) \times \left[2 + Q + \frac{1}{Q}\right] + 5\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right) \times \left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 2\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 40. \tag{15}$$

Lemma 4. *[3] If* $P = \dfrac{\varphi(q)\varphi(q^7)}{\varphi(q^9)\varphi(q^{63})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{63})}{\varphi(q^9)\varphi(q^7)}$*, then*

$$Q^4 + \frac{1}{Q^4} - 35\left(Q^3 + \frac{1}{Q^3}\right) - 413\left(Q^2 + \frac{1}{Q^2}\right) - 1379\left(Q + \frac{1}{Q}\right) - 1694 - \left(P^3 + \frac{9^3}{P^3}\right) - 7\left(P^2 + \frac{9^2}{P^2}\right)\left[7 + 3\left(Q + \frac{1}{Q}\right)\right] - 21\left(P + \frac{9}{P}\right) \left[21 + 14\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right)\right] + 7\left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 63\left[\sqrt{P} + \frac{3}{\sqrt{P}}\right]\left[7\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 14\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + \left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right)\right] + 21\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[2\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 7\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 0. \tag{16}$$

Lemma 5. *[5, p.56]*

$$\frac{f^3(q)}{qf^3(q^9)} = \frac{\varphi^2(q)}{\varphi^2(q^9)}\left\{\frac{\varphi(q) - 3\varphi(q^9)}{\varphi(q^9) - \varphi(q)}\right\}. \tag{17}$$

Lemma 6. *[1, Ch. 16, Entry 24(iii), p.39]*

$$\chi(q) = \frac{\varphi(q)}{f(q)}. \tag{18}$$

Lemma 7. *[4]*

$$I_{m,1} = 1 \tag{19}$$

Lemma 8. *[4]*

$$I_{m,n}I_{m,1/n} = 1 \tag{20}$$

3. General Theorems and Explicit Evaluations of $I_{m,n}$

Theorem 9. *If* $P := q\dfrac{\chi(q)\chi(q^2)}{\chi(q^9)\chi(q^{18})}$ *and* $Q := q^{-1/3}\dfrac{\chi(q)\chi(q^{18})}{\chi(q^9)\chi(q^2)}$ *then*

$$\left\{P^3+\frac{1}{P^3}\right\}+7\left\{P^2+\frac{1}{P^2}\right\}+14\left\{P+\frac{1}{P}\right\}-\left\{Q^3+\frac{1}{Q^3}\right\}+16$$
$$=\left\{\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right\}\left[\left\{\sqrt{P^5}+\frac{1}{\sqrt{P^5}}\right\}+4\left\{\sqrt{P^3}+\frac{1}{\sqrt{P^3}}\right\}+5\left\{\sqrt{P}+\frac{1}{\sqrt{P}}\right\}\right] \tag{21}$$

Proof. Replace q by q^9 in the lemma (6), we obtain

$$\chi(q^9)=\frac{\varphi(q^9)}{f(q^9)}. \tag{22}$$

Dividing the equations (18) by (22), we get

$$\frac{\chi(q)}{\chi(q^9)}=\frac{\varphi(q)}{\varphi(q^9)}\frac{f(q^9)}{f(q)}. \tag{23}$$

Raising the power three and also multiplying q on both side of the above equation (23), we get

$$q\frac{\chi^3(q)}{\chi^3(q^9)}=\frac{\varphi^3(q)}{\varphi^3(q^9)}\left\{q\frac{f^3(q^9)}{f^3(q)}\right\}. \tag{24}$$

Using the equation (17) and the above equation (24), we obtain

$$P^3+P(a-1)-3a=0 \tag{25}$$

where $P := \dfrac{\varphi(q)}{\varphi(q^9)}$, $a := q\dfrac{\chi^3(q)}{\chi^3(q^9)}$, solve the above equation we get, where $b = P^2$, solve the above equation we get

$$b=\frac{1-a+\sqrt{a^2+10a+1}}{2} \tag{26}$$

using the above equation (26) and equation (13), we obtain

$$\begin{aligned}&(y^3+x^3+4y^4x^7-16y^4x^4+4y^4x+y^5x^8-14y^5x^5+5y^5x^2+yx^7+4yx^4\\&+5y^2x^5+4y^7x^4+y^7x-y^7x^7-7y^2x^2+5x^6y^3-7x^6y^6-14x^3y^3+5x^3y^6\\&+y^8x^5-yx)(8y^5x^{14}+y^{16}x^{10}+y^{14}x^2+y^{14}x^{14}+9y^{14}x^8-y^{14}x^{11}-9y^8x^5\\&+8y^{14}x^5+5y^{13}x^{10}-5y^{13}x^{13}-3y^{13}x^7-5y^{13}x^4+6y^4x^7+27y^4x^4+y^4x\\&+6y^7x^{10}+y^6-3y^7x^{13}-9y^5x^8+y^{10}x^{16}-26y^5x^{11}-44y^5x^5-y^5x^2-y^{11}x^{14}\\&+15y^4x^{10}-5y^4x^{13}+9y^8x^{14}-4yx^7+yx^4+6y^{10}x^7+15y^{10}x^4-y^{10}x\\&+5y^{10}x^{13}+50y^{10}x^{10}+27y^8x^8-9y^8x^{11}+8y^2x^{11}-y^2x^5-yx^{10}+6y^7x^4\\&-4y^7x-33y^7x^7+y^2x^2+y^2x^{14}+9y^2x^8-3x^9y^3-33x^9y^9+6x^9y^{12}+9y^8x^2\\&-5x^{12}y^3+15x^{12}y^6+6x^{12}y^9+27x^{12}y^{12}+6x^9y^6-4x^{15}y^9+x^{12}y^{15}-9y^{11}x^8\\&+5x^6y^3+50x^6y^6+6x^6y^9+15x^6y^{12}-5x^3y^3+5x^3y^6-3x^3y^9-5x^3y^{12}\\&-x^6y^{15}-4x^9y^{15}-x^{15}y^6+x^{15}y^{12}+x^6-44y^{11}x^{11}-26y^{11}x^5+8y^{11}x^2)=0\end{aligned} \tag{27}$$

where $x = q^{1/3}\dfrac{\chi(q)}{\chi(q^9)}$ and $y = q^{2/3}\dfrac{\chi(q^2)}{\chi(q^{18})}$, by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof. □

Remark 1. Here by using the definition of (12), then above theorem (9) is also can be written as $P = I_{9,n}I_{9,4n}$ and $Q = \dfrac{I_{9,n}}{I_{9,4n}}$

Corollary 10. *We have*

$$I_{9,2} = \left[\frac{(\sqrt{3}+\sqrt{2})(3\sqrt{3}-5)}{\sqrt{2}}\right]^{1/3}, \tag{28}$$

$$I_{9,1/2} = \left[\frac{(\sqrt{3}-\sqrt{2})(3\sqrt{3}+5)}{\sqrt{2}}\right]^{1/3}, \tag{29}$$

$$I_{9,4} = \left[\frac{\sqrt{6}(1+\sqrt{3})-2-2\sqrt{(3+\sqrt{3})(\sqrt{3}-\sqrt{2})}}{4}\right], \tag{30}$$

$$I_{9,1/4} = \left[\frac{\sqrt{6}(1+\sqrt{3}) - 2 + 2\sqrt{(3+\sqrt{3})(\sqrt{3}-\sqrt{2})}}{4}\right]. \tag{31}$$

Proof. Setting $n = 1/2$ in Theorem (9) and using the Lemma (8), we obtain

$$A^{12} + 20A^9 - 60A^6 + 20A^3 + 1 = 0 \tag{32}$$

where $A := I_{9,n} < 1$, by solving the above equation, we obtain (28) and (29).

Again setting $n = 1$ in Theorem (9) and using Lemma (7), we get

$$B^8 + 4B^7 - 2B^6 - 8B^5 - 8B^4 - 8B^3 - 2B^2 + 4B + 1 = 0 \tag{33}$$

where $B := I_{9,4n} < 1$, the above equation can be written as

$$D^4 + 4D^3 - 6D^2 - 20D - 2 = 0 \tag{34}$$

where $D := B + \frac{1}{B}$, solving the above equation (34), we obtain (30) and (31). $\square$

Theorem 11. *If* $P := q^{4/3}\dfrac{\chi(q)\chi(q^3)}{\chi(q^9)\chi(q^{27})}$ *and* $Q := q^{-2/3}\dfrac{\chi(q)\chi(q^{27})}{\chi(q^9)\chi(q^3)}$ *then*

$$\begin{aligned}
&74 + \left\{P^6 + \frac{1}{P^6}\right\} + 35\left\{Q^3 + \frac{1}{Q^3}\right\} + 8\left\{Q^6 + \frac{1}{Q^6}\right\} - \left\{Q^9 + \frac{1}{Q^9}\right\} \\
&+ 7\left\{P^3 + \frac{1}{P^3}\right\}\left[28 + 19\left\{Q^3 + \frac{1}{Q^3}\right\}\right] + \left\{Q^6 + \frac{1}{Q^6}\right\}\left[28 + 19\left\{P^3 + \frac{1}{P^3}\right\}\right] \\
&+ 10\left\{\sqrt{P^9} + \frac{1}{\sqrt{P^9}}\right\}\left\{\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right\} = 10\left\{\sqrt{P^3} + \frac{1}{\sqrt{P^3}}\right\}\left[\left\{\sqrt{Q^{15}} + \frac{1}{\sqrt{Q^{15}}}\right\}\right. \\
&\left. -17\left\{\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right\} - 46\left\{\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right\}\right].
\end{aligned} \tag{35}$$

Proof. Employing the equations (26) and equation (14), we obtain

$$
\begin{aligned}
&10x^6y^3 - x^{18}y^6 + 35x^{12}y^6 + 19y^{15}x^9 + 8y^{15}x^3 + 17x^{12}y^3 + 10x^{15}y^{12}\\
&+ 10x^{12}y^{15} - y^{18}x^3 + x^3y^3 + x^{15}y^{15} - x^{18}y^3 - x^6y^{18} + 19x^3y^9 - x^{15}\\
&- y^{12} - x^{18} - y^{18} + 17y^{15}x^6 + 17y^{12}x^3 + 28x^{12}y^{12} + 28x^6y^6 + 10x^3y^6\\
&+ 74x^9y^9 + 35x^6y^{12} + 46x^9y^{12} + 46x^6y^9 + 19x^9y^3 + 19x^{15}y^9 + 8x^{15}y^3\\
&+ 46x^{12}y^9 + 17x^{15}y^6 - x^{12} - y^{15} + 46x^9y^6 = 0
\end{aligned} \tag{36}
$$

where $x = q^{1/3}\dfrac{\chi(q)}{\chi(q^9)}$ and $y = q\dfrac{\chi(q^3)}{\chi(q^{27})}$. This completes the proof. □

Remark 2. Here by using the definition of (12), then above theorem (11) is also can be written as $P = I_{9,n}I_{9,9n}$ and $Q = \dfrac{I_{9,n}}{I_{9,9n}}$

Corollary 12. *We have*

$$I_{9,3} = \left[2^{1/3} - 1\right]^{1/3}, \tag{37}$$

$$I_{9,1/3} = \left[2^{2/3} + 2^{1/3} + 1\right]^{1/3}, \tag{38}$$

$$I_{9,9} = \left[\frac{u^2 + 3u + 13 - \sqrt{u^4 + 34u^2 + 78u + 451 + 12\sqrt{3}}}{u}\right]^{1/3}, \tag{39}$$

$$I_{9,1/9} = \left[\frac{u^2 + 3u + 13 + \sqrt{u^4 + 34u^2 + 78u + 451 + 12\sqrt{3}}}{u}\right]^{1/3}. \tag{40}$$

where $u := (47 + 2\sqrt{3})^{1/3}$

Proof. Setting $n = 1/3$ in Theorem (4) and using the Lemma (8), we obtain

$$
\begin{aligned}
(A^9 + 3A^6 + 3A^3 - 1)(A^9 - 3A^6 - 3A^3 - 1)(A + 1)^2(A^2 - A + 1)^2(A^2 + 1)^2\\
(A^4 - A^2 + 1)^2 = 0,
\end{aligned} \tag{41}
$$

where $A := I_{9,n} < 1$, by solving the first factor we obtain (37) and similarly by solving the second factor we get (38), the other factors does not satisfying the condition of $I_{9,9} < 1$.

Again setting $n = 1$ in Theorem (11) and using Lemma (7), we obtain

$$54B^{15}+135B^{12}-3B^{18}+135B^{6}+54B^{3}+204B-3=0 \tag{42}$$

where $B := I_{9,9n} < 1$, the above equation can be written as

$$3M^{3}-54M^{2}-144M-96=0 \tag{43}$$

where $M := B^3 + \dfrac{1}{B^3}$, solving the above equation (43), we get (39) and (40) □

Theorem 13. *If* $P := q^{2}\dfrac{\chi(q)\chi(q^{5})}{\chi(q^{9})\chi(q^{45})}$ *and* $Q := q^{-4/3}\dfrac{\chi(q)\chi(q^{45})}{\chi(q^{9})\chi(q^{5})}$ *then*

$$10+5\left\{P+\frac{1}{P}\right\}+\left\{Q^{3}+\frac{1}{Q^{3}}\right\}=5\left\{\sqrt{P}+\frac{1}{\sqrt{P}}\right\}\left\{\sqrt{Q^{3}}+\frac{1}{\sqrt{Q^{3}}}\right\}. \tag{44}$$

Proof. Employing the equation (26) and equation (15), we obtain (44) □

Remark 3. Here by using the definition of (12), then above theorem (13) is also can be written as $P = I_{9,n}I_{9,25n}$ and $Q = \dfrac{I_{9,n}}{I_{9,25n}}$

Corollary 14. *We have*

$$I_{9,5}=\left[4-\sqrt{15}\right]^{1/3}, \tag{45}$$

$$I_{9,1/5}=\left[4+\sqrt{15}\right]^{1/3}. \tag{46}$$

$$I_{9,25}=\frac{2+\sqrt{15}-\sqrt{4\sqrt{15}+15}}{2}. \tag{47}$$

$$I_{5,1/25}=\frac{2+\sqrt{15}+\sqrt{4\sqrt{15}+15}}{2}. \tag{48}$$

Proof. Employing Theorem (13), Lemma (8) and (7), we arrive (45)-(48) □

Theorem 15. *If* $P := q^{8/3}\dfrac{\chi(q)\chi(q^{7})}{\chi(q^{9})\chi(q^{63})}$ *and* $Q := q^{-2}\dfrac{\chi(q)\chi(q^{63})}{\chi(q^{9})\chi(q^{7})}$ *then*

$$\begin{aligned}&\left\{Q^{4}+\frac{1}{Q^{4}}\right\}-14\left\{Q^{2}+\frac{1}{Q^{2}}\right\}-35\left\{Q+\frac{1}{Q}\right\}-\left\{P^{3}+\frac{1}{P^{3}}\right\}\\ &7\left\{\sqrt{Q^{3}}+\frac{1}{\sqrt{Q^{3}}}\right\}\left[\left\{\sqrt{P^{3}}+\frac{1}{\sqrt{P^{3}}}\right\}+3\left\{\sqrt{P}+\frac{1}{\sqrt{P}}\right\}\right].\end{aligned} \tag{49}$$

Proof. Employing the equations (26) and equation (16), we obtain (49). □

Remark 4. Here by using the definition of (12), then above theorem (13) is also can be written as $P = I_{9,n}I_{9,49n}$ and $Q = \frac{I_{9,n}}{I_{9,49n}}$

Corollary 16. *We have*

$$I_{9,7} = \frac{1+\sqrt{21}-\sqrt{6+2\sqrt{21}}}{4}. \tag{50}$$

$$I_{9,1/7} = \frac{1+\sqrt{21}+\sqrt{6+2\sqrt{21}}}{4} \tag{51}$$

$$I_{9,49} = \frac{1}{4}\left[4+3\sqrt{7}+\sqrt{X}-\sqrt{2X+8\sqrt{X}+6\sqrt{7X}}\right]. \tag{52}$$

$$I_{9,1/49} = \frac{1}{4}\left[4+3\sqrt{7}+\sqrt{X}+\sqrt{2X+8\sqrt{X}+6\sqrt{7X}}\right]. \tag{53}$$

where $X = 63 + 24\sqrt{7}$.

Proof. Employing Theorem (15), Lemma (8) and (7), we arrive (50)-(53) □

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