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ON THE DIOPHANTINE EQUATION $5^x + 7^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In 1844, Catalan [2] conjectured that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

The Catalan's conjecture was proved by Mihailescu [5] in 2004.

Many Diophantine equations of type $a^x + b^y = z^2$ were solved by the Catalan's conjecture (see [1, 3, 4, 7, 8, 9, 10, 11, 12]).

In 2012, we [6] showed that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers, and then we posed an open problem that, for any positive odd prime numbers p and q such that $q - p = 2$, what is the set of all solutions (x, y, z) for the Diophantine equation $p^x + q^y = z^2$ where x, y and z are non-negative integers.

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In this paper, we will use the Catalan's conjecture to show that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [5] *$(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [6] *The Diophantine equation $5^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers.*

Lemma 2.3. *The Diophantine equation $1 + 7^y = z^2$ has no non-negative integer solution where y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 7^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 7^y \geq 1 + 7^1 = 8$. Then $z \geq 3$. Now, we consider on the equation $z^2 - 7^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 8$. This is a contradiction. Hence, the equation $1 + 7^y = z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. *The Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $5^x + 7^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 4$. Moreover, $5^x \equiv 1 \pmod 4$. This implies that $7^y \equiv 3 \pmod 4$. Thus, y is odd. It follows that $7^y \equiv 2 \pmod 5$ or $7^y \equiv 3 \pmod 5$. We obtain that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This is impossible. □

Corollary 3.2. *The Diophantine equation $5^x + 7^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $5^x + 7^y = w^4$. Let $z = w^2$. Then $5^x + 7^y = z^2$. By Theorem 3.1, the equation

$5^x + 7^y = z^2$ has no non-negative integer solution. This implies that the equation $5^x + 7^y = w^4$ has no non-negative integer solution. □

Corollary 3.3. *The Diophantine equation $25^u + 49^v = z^2$ has no non-negative integer solution where u, v and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, v and z such that $5^u + 7^v = w^4$. Let $x = 2u$ and $y = 2v$. Then $5^x + 7^y = z^2$. By Theorem 3.1, the equation $5^x + 7^y = z^2$ has no non-negative integer solution. This implies that the equation $25^u + 49^v = z^2$ has no non-negative integer solution. □

4. Open Problem

In [6], $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In this paper, the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. We also note that 3, 5 and 7 are odd prime numbers such that $5 - 3 = 2$ and $7 - 3 = 2$. We may ask for the set of all solutions (x, y, z) for the Diophantine equation $3^x + 5^y + 7^z = w^2$ where x, y, z and w are non-negative integers.

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