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MORE ON THE DIOPHANTINE EQUATION $2^x + 37^y = z^2$

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Abstract: In this paper, we show that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 37^y = z^2$ where x, y and z are non-negative integers.

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1. Introduction

In 2007, Acu [1] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are only two non-negative integer solutions for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2013, Sroysang [9] showed that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. In the same year, Chotchaisthit [2] showed that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers. Moreover, Chotchaisthit [2] also posed an open problem that what is the set of all non-negative integer solutions of $2^x + 37^y = z^2$ where x, y and z are non-negative integers. In this paper, we show that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 37^y = z^2$ where x, y and z are non-negative integers. For the solutions of some exponential Diophantine

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equations, we refer to [3, 5, 6, 7, 8, 10, 11].

2. Preliminaries

Proposition 2.1. [4] $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [1] $(3, 3)$ *is a unique solution* (x, z) *for the Diophantine equation* $2^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1 + 37^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 37^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 37^y \geq 1 + 37^1 = 38$. Then $z \geq 7$. Now, we consider on the equation $z^2 - 37^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 38$. This is a contradiction. Hence, the equation $1 + 37^y = z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. $(3, 0, 3)$ *is a unique solution* (x, y, z) *for the Diophantine equation* $2^x + 37^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $2^x + 37^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Thus, z is odd. Then there is a non-negative integer t such that $z = 2t + 1$. We obtain that $2^x + 37^y = 4(t^2 + t) + 1$. Then $37^y \equiv 1 \pmod 4$. Thus, y is even. Then there is a non-negative integer k such that $y = 2k$. We divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 3$ and $z = 3$.

Case $y \geq 2$. Then $k \geq 1$. Then $z^2 - 37^{2k} = 2^x$. Then $(z - 37^k)(z + 37^k) = 2^x$. We obtain that $z - 37^k = 2^u$ where u is a non-negative integer. Then $z + 37^k = 2^{x-u}$. It follows that $2(37^k) = 2^{x-u} - 2^u = 2^u(2^{x-2u} - 1)$. We divide the number u into two subcases.

Subcase $u = 0$. Then $z - 37^k = 1$. Thus, z is even. This is a contradiction.

Subcase $u = 1$. Then $2^{x-2} - 1 = 37^k$. It follows that $2^{x-2} = 37^k + 1 \geq 37 + 1 = 38$. Thus, $x \geq 3$. Moreover, $2^{x-2} - 37^k = 1$. By Proposition 2.1, we have $k = 1$. Then $2^{x-2} = 38$. This is impossible.

Therefore, $(3, 0, 3)$ is a unique solution (x, y, z) for the equation $2^x + 37^y = z^2$. □

Corollary 3.2. *The Diophantine equation $2^x + 37^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $2^x + 37^y = w^4$. Let $z = w^2$. Then $2^x + 37^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 0, 3)$. Then $w^2 = z = 3$. This is a contradiction. □

Corollary 3.3. $(1, 0, 3)$ *is a unique solution (u, y, z) for the Diophantine equation $8^u + 37^y = z^2$ where u, y and z are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $8^u + 37^y = z^2$. Let $x = 3u$. Then $2^x + 37^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 0, 3)$. Then $3u = x = 3$. Thus, $u = 1$. Therefore, $(1, 0, 3)$ is a unique solution (u, y, z) for the equation $8^u + 37^y = z^2$. □

Corollary 3.4. *The Diophantine equation $32^u + 37^y = z^2$ has no non-negative integer solution where u, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, y and z such that $32^u + 37^y = z^2$. Let $x = 5u$. Then $2^x + 37^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 0, 3)$. Then $5u = x = 3$. This is a contradiction. □

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