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CONNECTEDNESS BY CONTINUOUS FUNCTIONS AND TEST SPACES

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Abstract: The concept of connectedness of a set is characterized using continuous functions from the set to a Connectedness Test Space. The Connectedness Test Space is defined using the connected subsets of the set of Real Numbers, as a model. Several classical results of connectedness are proved using this approach.

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1. Introduction

One of the important notions encountered in topology courses is connectedness (see [1], [7]). Recall that a topological space is defined to be *connected* if it is not the union of two nonempty disjoint open subsets. A subset of a topological space is connected if it is connected when it has the the relative topology. An excellent history of the development and acceptance of this definition of connectedness is given in [8].

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The following proposition gives characterizations (scattered throughout the literature) of connectedness in terms of certain continuous functions into topological spaces with a known collection of connected subsets.

Proposition 1. *Each of the following statements is equivalent to a space X being connected.*

1. *Each continuous function from X to the reals with the discrete topology is constant [7].*

2. *Each continuous function from X to the Euclidean line has an interval as its image [6].*

3. *Each nonconstant continuous function from X to the Euclidean line with the cofinite topology has an infinite image [1].*

In connection with the characterizations in Proposition 1, the following observations are made.

It is easy to see that if the real line has the discrete topology, the connected sets are precisely the emptyset and the singletons.

While, historically, the notion of 'connected subsets' of a general topological space was not directly motivated by properties of intervals on the Euclidean line, such a motivation is presented here. A subset I of the reals is an *interval in the reals* if $z \in I$ whenever $x, y \in I$ and $x < z < y$. This definition has the virtue of encompassing all open, closed, half-open, half-closed, intervals and half-lines. If I is a nonempty interval with a lower bound (an upper bound), the infimum (supremum) of I is called the *left endpoint (right endpoint)* of I. An interval in the reals will be simply referred to as an 'interval'. A nonempty bounded open interval is one with two real numbers as endpoints but which has neither endpoint as an element. It is well-known that the open subsets of the Euclidean line may be described as a collection consisting of the emptyset and unions of nonempty bounded open intervals. It is easy to prove that intervals in the reals have the following properties:

1. The emptyset, $\emptyset$, and singleton subsets are intervals.

2. An interval with more than one point is uncountable.

3. If θ is any collection of intervals and $\bigcap_{I\in\theta} I \neq \emptyset$ then $\bigcup_{I\in\theta} I$ is an interval.

4. If J is an interval and $J \subset K \subset \overline{J}$ then K is an interval (where $\overline{J}$ is J along with its endpoints).

5. The connected subsets on the Euclidean line are precisely the intervals.

If the open subsets of the real line are the emptyset and the complements of finite sets (the so-called "cofinite topology" on the reals), it can be established that the connected subsets are precisely the emptyset, singleton sets, and infinite sets.

Hence the requirement in each of the statements in Proposition 1 is that a real valued continuous function on the space has a connected image. It is easy to see that singleton sets in each of these spaces are closed subsets and it can be shown directly in these particular cases that the collection of connected subsets satisfies the following properties:

Properties.

1. If θ is any collection of connected sets and $\bigcap_{M\in\theta} M \neq \emptyset$ then $\bigcup_{M\in\theta} M$ is connected.
2. If M is a connected set and $M \subset N \subset \overline{M}$ then N is connected (Where $\overline{M}$ is the closure of M in the space).

2. Development, Characterizations and Results

In view of the above observations, abstracting the result that a space is connected if and only if each continuous real valued function on the space has connected image, we introduce the concept of a *connectedness test space*. The connectedness test space, T, is then used to characterize connectedness of any space. Varying the space T produces information about connected spaces and new proofs of classical results, as is demonstrated in Theorems 2.2-2.7 below. Theorem 2.1 establishes the existence of such a connectedness test space. This approach also provides a characterization (Theorem 2.8) of connectedness of a space in terms of the support of continuous functions on the space, where by the *support of a function* we mean the set $\{x \in X | f(x) \neq 0\}$, and f is a real valued function.

A topological space T with more than one point is called a *connectedness test space* if singletons are closed subsets and the collection of connected subsets C_T of the space has been determined and shown directly to satisfy the properties (1) and (2) above. In this paper, it is first shown that a connected subset M of an arbitrary topological space is characterized by continuous functions on M

mapping into any test space. That is, a connected subset of a space is defined as follows:

A subset M of a topological space X is connected if and only if $f(M)$ is connected where f is a continuous function from X to a connectedness test space.

Then, some demonstrations of the flavor of proofs of some well-known general properties of connected subsets of a space derived from this characterization are offered. Finally, the characterization is utilized to provide a proof of an old theorem due to Kline [4], providing a new proof each time the connectedness test space is changed.

Recall that a topological space is *totally disconnected* if $\emptyset$ and the singletons are the only connected subsets of the space [7].

(Kline) *If a topological space X is connected and there is an $x \in X$ such that $X - \{x\}$ is totally disconnected, then $X - \{y\}$ is connected for each $y \in X - \{x\}$.*

In the sequel T will be a connectedness test space where C_T, its collection of connected subsets, has been determined.

Theorem 2.1. *The following statements are equivalent for a topological space X.*

1. *The space X is connected.*
2. *Each continuous $f : X \to T$ satisfies $f(X) \in C_T$ for each test space T.*
3. *Each continuous closed $f : X \to T$ satisfies $f(X) \in C_T$ for each test space T.*
4. *There is a T such that each continuous closed $f : X \to T$ satisfies $f(X) \in C_T$.*
5. *There is a T such that each continuous $f : X \to T$ satisfies $f(X) \in C_T$.*

Proof. (1) $\Longrightarrow$ (2). The continuous image of a connected space is connected, by definition.

(2) $\Longrightarrow$ (3), (5) $\Longrightarrow$ (4). Obvious.

(2) $\Longrightarrow$ (5), (3) $\Longrightarrow$ (4). Let T be the Euclidean line.

(4) $\Longrightarrow$ (1). If M, N are nonempty, open, disjoint subsets of X with $X = M \cup N$, choose any test space T, $p, q \in T$, $p \neq q$. Then $f : X \to T$ defined by

$$f(x) = \begin{cases} p & \text{if } x \in M \\ q & \text{if } x \in N \end{cases}$$

is continuous and closed but $f(X) = \{p, q\} \notin C_T$. □

Some applications of Theorem 2.1 are now given where T is the Euclidean line. The Euclidean line is represented by $\Re$. A real valued function f on X has the *Darboux property* [6] if $z \in f(X)$ whenever $f(x) < z < f(y)$ for some x, y. Corollary 2.1 is now obvious.

Corollary 2.1. *A topological space X is connected if and only if each continuous $f : X \to \Re$ has the Darboux property.*

Corollary 2.2. *If X is a countable connected space, then every continuous real valued function on X is constant.*

Proof. If a continuous $f : X \to \Re$ is not constant, then $f(X)$ is an interval with more than one point and is consequently uncountable. Hence X must be uncountable. □

Corollary 2.3. *A countable connected regular space is not completely regular.*

Proof. Immediate, in view of Corollary 2.2. □

Corollary 2.4. *A Hausdorff connected normal topological space with more than one point must be uncountable.*

Proof. In view of the Urysohn's Lemma, there are nonconstant real-valued continuous functions on the space. □

Corollaries 2.5 and 2.6 are applications of Theorem 2.1(5), where T is the cofinite topology on the reals.

Corollary 2.5. *The closed interval $[0, 1]$ is a connected subset of the Euclidean line.*

Proof. Let $f : [0,1] \to T$ be continuous, nonconstant and suppose $f([0,1])$ is finite. Then there exist $x, y \in [0,1], x < y$ satisfying $f([x,y])$ is finite and $f(x) \neq f(y)$. Let $A = f^{-1}(f([x,y]) - \{f(y)\}) \cap [x,y]$, $B = f^{-1}(\{f(y)\}) \cap [x,y]$. Then A, B are closed, disjoint, nonempty subsets of $[0,1]$. Let $z = \sup A$. Then $z \in A$, $x \le z < y$. However this implies that $f(z) = f(y)$, contradicting $A \cap B = \emptyset$. Hence $f([0,1])$ is infinite. □

Corollary 2.6. *Complements of countable subsets of the Euclidean plane are connected.*

Proof. Let X be the Euclidean plane, let $Q \subset X$ be countable, let $f : X - Q \to T$ be continuous, and let $p, q \in X - Q$ with $f(p) \neq f(q)$. There are uncountably many $z \in X$ such that $||p - z|| = ||q - z||$, where $|| \cdot ||$ is the Euclidean norm. Geometrically, the collection of such z is the perpendicular bisector of the line segment joining p to q. For each such z define $f_z : [0,1] \to T$ by

$$f_z(r) = \begin{cases} 2rz + (1-2r)p & \text{if} \quad 0 \le r \le 1/2 \\ 2(1-r)z + (2r-1)q & \text{if} \quad 1/2 \le r \le 1. \end{cases}$$

Then f_z is continuous and, since $f_z([0,1]) \cap f_w([0,1]) = \{p, q\}$ when $z \neq w$, there are uncountably many f_z. Since Q is countable,there is a z such that $f_z([0,1]) \subset X - Q$. Choose such a z and consider the composition $f \circ f_z$. Then $f \circ f_z([0,1])$ is infinite, since $f \circ f_z$ is continuous, nonconstant, and $[0,1]$ is a connected subset of the Euclidean line; hence $f(X - Q)$ is infinite. Thus, $X - Q$ is connected. □

In the remainder of this article, T is any connectedness test space. Next, some well-known properties of connectedness are proved using the characterization in Theorem 2.1.

Theorem 2.2. *If M is a connected subspace of a topological space X and $M \subset N \subset \overline{M}$ then N is a connected subspace of X.*

Proof. Let $f : N \to T$ be continuous. Then f is continuous on M and $f(M) \in C_T$. Also $f(M) \subset f(N) = f(N \cap \overline{M}) \subset \overline{f(M)}$, so $f(N) \in C_T$. □

Theorem 2.3. *If θ is a collection of connected subspaces of a topological space X and $\bigcap_\theta M \neq \emptyset$, then $\bigcup_\theta M$ is a connected subspace of X.*

Proof. Let $f : \bigcup_\theta M \to T$ be continuous. Then f is continuous on each $M \in \theta$ and consequently $f(M) \in C_T$. Furthermore $\emptyset \neq f\left(\bigcap_\theta M\right) \subset \bigcap_\theta f(M)$ and $\bigcup_\theta f(M) = f\left(\bigcup_\theta M\right)$. Hence $f\left(\bigcup_\theta M\right) \in C_T$. □

If the definition of connected topological spaces in terms of continuous functions and test spaces is taken as the basic definition, the following interesting proof that connectedness is a continuity invariant may be given.

Theorem 2.4. *If a topological space X is connected and $f : X \to Y$ is continuous, then $f(X)$ is a connected subspace of Y.*

Proof. If $g : f(X) \to T$ is continuous, then $g \circ f : X \to T$ is continuous. Hence $g(f(X)) = (g \circ f)(X) \in C_T$. □

Theorem 2.5. *If X, Y are connected topological spaces, then the product space $X \times Y$ is connected.*

Proof. If $(p, q) \in X \times Y$ the function $h : X \to X \times \{q\}$ $(respectively, h : Y \to \{p\} \times Y)$ defined by $h(x) = (x, q)$ $(respectively, h(y) = (p, y))$ is continuous (in fact, a homeomorphism). So $X \times \{q\}$ $(respectively, \{p\} \times Y)$ is connected in view of Theorem 2.4. Fix $(p, q) \in X \times Y$. For each $(a, b) \in X \times Y$ let $C(a, b) = (X \times \{b\}) \cup (\{p\} \times Y)$. Then $(a, b) \in C(a, b)$, $\{(p, q)\} \subseteq \bigcap_{(a,b) \in X \times Y} C(a, b)$, $C(a, b)$ connected, so $X \times Y = \bigcup_{(a,b) \in X \times Y} C(a, b)$ is connected in view of Theorem 2. 3. □

Since the topological structure of a product space is preserved under associativity, Theorem 2.5 and induction lead to

Proposition 2. *The product of a finite collection of connected topological spaces is connected.*

Theorem 2.6 gives a proof, using the definition of connectedness of a space in terms of continuous functions and test space method introduced here, of the well-known fact that Theorem 2.5 extends to arbitrary products.

Theorem 2.6. *If $\{X_\alpha\}_\Sigma$ is a family of connected topological spaces, then the product space $X = \Pi_\Sigma X_\alpha$ is connected.*

Proof. Let $X = \Pi_\Sigma X_\alpha$ be the product and let $f : X \to T$ be a continuous, closed function. Choose $y \in X$ and let Ω be the collection of connected subsets of X which have y as an element. Then $\bigcup_\Omega M$ is connected by Theorem 2.3, and $\overline{f\left(\bigcup_\Omega M\right)}$ is connected by Theorems 2.2 and 2.4. Now $\overline{f\left(\bigcup_\Omega M\right)} \subset f(X)$ since f is continuous and closed. Suppose $f(x) \in f(X) - \overline{f\left(\bigcup_\Omega M\right)}$, and let V be open in T such that $f(x) \in V$, $V \cap f\left(\bigcup_\Omega M\right) = \emptyset$. Choose a finite $\Gamma \subset \Sigma$ and, for each $\alpha \in \Gamma$, an open set A_α in X_α such that $x \in \bigcap_\Gamma {\pi_\alpha}^{-1}(A_\alpha) \subset f^{-1}(V)$, where π_α is the α-projection of X onto X_α. Define $z \in X$ by its coordinates as follows; $z_\alpha = x_\alpha$ if $\alpha \in \Gamma$, $z_\alpha = y_\alpha$ if $\alpha \in \Sigma - \Gamma$. Then $y, z \in M_y = \{p \in X : p_\alpha = y_\alpha \text{ for } \alpha \in \Sigma - \Gamma\}$, which is homeomorphic to $\Pi_\Gamma X_\alpha$, and consequently connected by Proposition 2. This is a contradiction since $f(z) \in V$, $f(M_y) \subset \overline{f\left(\bigcup_\Omega M\right)}$. Hence $f(X)$ is connected. □

In 1921 Knaster and Kuratowski [5] constructed an example of a connected space having the property that the removal of a single point leaves the resulting space totally disconnected. Motivated by this example, Kline [4] conjectured and proved Theorem 2.7. As another illustration of the new approach developed here, a new proof of this theorem is given.

Theorem 2.7. *Let X be a connected space and $x \in X$ so that $X - \{x\}$ is totally disconnected. Then $X - \{y\}$ is connected for each $y \in X - \{x\}$.*

Proof. Lemmas 2.1 and 2.2, due to Jackson [3], will be utilized in the proof of the theorem. Proofs are included here for the sake of completeness. □

Lemma 2.1. *Let X be a topological space and $x \in X$. Then $A \subset X - \{x\}$ is open (closed) in $X - \{x\}$ if and only if A or $A \cup \{x\}$ is open (closed) in X.*

Proof. If A is open (closed) in X, then A is open (closed) in $X - \{x\}$ since $(X - \{x\}) \cap A = A$. If $A \cup \{x\}$ is open (closed) in X, then A is open (closed) in $X - \{x\}$ since $(X - \{x\}) \cap (A \cup \{x\}) = A$. The sufficiency is proved. As for the necessity, if A is open (closed) in $X - \{x\}$, Then $A = (X - \{x\}) \cap Q$ where Q is open (closed) in X. Then

$$\begin{cases} A = Q & \text{if } x \notin Q \\ A \cup \{x\} = Q & \text{if } x \in Q. \end{cases}$$

Hence A or $A \cup \{x\}$ is open (closed) in X. □

Lemma 2.2. *Let X be a connected topological space and $x \in X$. Let A be nonempty, open and closed in $X - \{x\}$ and $A \neq X - \{x\}$. Then A is open (closed) in X if and only if $A \cup \{x\}$ is closed (open) in X.*

Proof. For the sufficiency, if $A \cup \{x\}$ is closed (open) in X, then $A \cup \{x\}$ is not open (closed) in X since X is connected and $A \neq X - \{x\}$ implies $A \cup \{x\} \neq X$. From Lemma 1, A is open (closed) in X. To the necessity, if A is open (closed) in X, then A is not closed (open) in X since $A \neq \emptyset$, $A \neq X$. From Lemma 1, $A \cup \{x\}$ is closed (open) in X. □

Completing the Proof of Theorem 2.7. Let $y \in X - \{x\}$, let $f : X - \{y\} \to T$ be continuous and suppose M, N are disjoint, nonempty, subsets open in $f(X - \{y\})$ such that $f(X - \{y\}) = M \cup N$. Suppose $f(x) \notin M$. Let $Z = f^{-1}(M) \cup \{y\}$. Then Z has more than one element and $Z \subset X - \{x\}$, so Z is not connected. Let $Z = H \cup K$, H, K, open in Z, $H \cap K = \emptyset$, $H \neq \emptyset, K \neq \emptyset$. Suppose that $y \notin H$. Let $S = K \cup f^{-1}(N)$. Then $H \cup S = X$, $H \cap S = \emptyset$, $H \neq \emptyset$, $S \neq \emptyset$. Since $Z - \{y\} = f^{-1}(M)$ is a nonempty open and closed proper subset of $X - \{y\}$, Lemma 2.2 leads to the following two possibilities:

$$\begin{cases} (1)\ Z - \{y\} & \text{is open in } X \text{ and } Z \text{ is closed in } X \\ (2)\ Z - \{y\} & \text{is closed in } X \text{ and } Z \text{ is open in } X. \end{cases}$$

Under (1), H is open in X since H is open in Z and $y \notin H$; and H is closed in X since H is closed in Z which is closed in X. Under (2), H is open in X since H is open in Z which is open in X; and H is closed in X since H is closed in Z and $y \notin H$; this contradicts the connectedness of X. Therefore, $X - \{y\}$ is connected for each $y \in X - \{x\}$. □

Since the connectedness of a space X is characterized using continuous function from X to a *connectedness test space* T and the collection C_T of connected subsets of T, the above proof of Theorem 2.7 due to Kline[4], provides numerous new proofs. One only needs to change the topology of the range space. Note that in the proof of Theorem 2.7, T is any connectedness test space.

Remark. By characterizing connectedness through continuous functions, we have provided a new approach to the study of this concept. It is instructional to observe that the connectedness of a space is determined solely by its image under a continuous function into a connectedness test space and such a space always exists. The line of proofs of Corollaries 2.5 and 2.6, Theorems 2.6 and 2.7, just to mention a few, are especially noteworthy, in this respect.

In addition, we have demonstrated that the Euclidean Line is a connectedness test space. It is well known that the removal of a single point makes the remainder of the Euclidean Line disconnected. That is, each point of the Euclidean Line is a cut point. Considering these facts, the approach provided here provides a generalized approach of studying the concept of connectedness using the notion of the support of continuous functions on the space. With this observation the following result is immediate.

Theorem 2.8. *A space X is connected if and only if the support of every real-valued continuous function on the space X is disconnected.*

Observations. Note that if $f : X \to Y$ then (i) if X has the discrete topology, f is continuous and (ii) if Y has indiscrete topology, then f is continuous. Also, if Y has indiscrete topology and has more than one point, then it cannot be a connectedness test space.

Consider the identity function $f : (X, T) \to (X, T_1)$ where T_1 is the usual topology; T is the cofinite topology and X is the Euclidean Line. For each $a \in X, X - \{a\}$ is disconnected in T_1, but connected in T. Note that f is not continuous. Also both (X, T) and (X, T_1) can be used as a connectedness test space.

Let $p \in X$ and let the topology T above be the p -inclusion topology. Then the connected subsets of X are the empty set, singleton sets and any set which contains p. Note that if $p \notin (a, b)$ then (a, b) is connected in T_1, but not connected in T. It is to be observed that f is not continuous. Also, in this case (X, T) cannot be a connectedness test space since $\{p\}$ is not closed.

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