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GENERALIZED CAUCHY-SCHWARZ INEQUALITY

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Abstract: A generalized Cauchy-Schwarz inequality for more than two vectors is given.

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The Cauchy-Schwarz Inequality

$$(\mathbf{a}\cdot\mathbf{b})^2 \leq (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})$$

for $\mathbf{a},\mathbf{b}\in\mathbf{E}^n$ is generalized for more than two vectors in any real inner product space.

Theorem 1. *Let $\mathbf{v}_1,\ldots,\mathbf{v}_m$ be vectors in a real vector space with inner product $\langle\cdot,\cdot\rangle$. Then*

$$\sum_{p\in S_m-A_m}\prod_{i}^{m}\langle\mathbf{v}_i,\mathbf{v}_{p(i)}\rangle \leq \sum_{p\in A_m}\prod_{i}^{m}\langle\mathbf{v}_i,\mathbf{v}_{p(i)}\rangle \tag{1}$$

where S_m is the symmetric group and A_m the alternating group.

Proof. Let B be the $m\times m$ matrix with $B_{ij}=\langle\mathbf{v}_i,\mathbf{v}_j\rangle$, i.e.

$$B=\begin{pmatrix}\langle\mathbf{v}_1,\mathbf{v}_1\rangle & \cdots & \langle\mathbf{v}_1,\mathbf{v}_m\rangle\\ \vdots & \vdots & \vdots\\ \langle\mathbf{v}_m,\mathbf{v}_1\rangle & \cdots & \langle\mathbf{v}_m,\mathbf{v}_m\rangle\end{pmatrix} \tag{2}$$

B is symmetric, positive semi-definite and $det(B)\geq 0$. Since

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$$det(B) = \sum_{p \in S_m} sign(p) \prod_{i}^{m} \langle \mathbf{v}_i, \mathbf{v}_{p(i)} \rangle \tag{3}$$

where $sign(p) = 1$ for p even and $sign(p) = -1$ for p odd, the inequality follows.

If the inner product is the dot product, we have

$$\sum_{p \in S_m - A_m} \prod_{i}^{m} \sum_{k}^{n} \mathbf{v}_{ik} \mathbf{v}_{p(i)k} \leq \sum_{p \in A_m} \prod_{i}^{m} \sum_{k}^{n} \mathbf{v}_{ik} \mathbf{v}_{p(i)k} \tag{4}$$

Remark. For $m = 3$, we have

$$(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_3)^2 + (\mathbf{v}_2 \cdot \mathbf{v}_2)(\mathbf{v}_3 \cdot \mathbf{v}_1)^2 + (\mathbf{v}_3 \cdot \mathbf{v}_3)(\mathbf{v}_1 \cdot \mathbf{v}_2)^2$$
$$\leq (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2)(\mathbf{v}_3 \cdot \mathbf{v}_3) + 2(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{v}_3)(\mathbf{v}_3 \cdot \mathbf{v}_1)$$

Rearranging the terms, we have

Theorem 2. *If $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are nonzero vectors in $\mathbf{E}^n$, then*

$$\frac{(\mathbf{v}_1 \cdot \mathbf{v}_2)^2}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2)} + \frac{(\mathbf{v}_2 \cdot \mathbf{v}_3)^2}{(\mathbf{v}_2 \cdot \mathbf{v}_2)(\mathbf{v}_3 \cdot \mathbf{v}_3)} + \frac{(\mathbf{v}_3 \cdot \mathbf{v}_1)^2}{(\mathbf{v}_3 \cdot \mathbf{v}_3)(\mathbf{v}_1 \cdot \mathbf{v}_1)}$$
$$\leq \quad 1 + 2\frac{(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{v}_3)(\mathbf{v}_3 \cdot \mathbf{v}_1)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2)(\mathbf{v}_3 \cdot \mathbf{v}_3)}$$

The equality holds if and only if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

Corollary 1. *If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors, then*

$$(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{b} \cdot \mathbf{c})^2 + (\mathbf{c} \cdot \mathbf{a})^2 \leq 1 + 2(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{a}) \tag{5}$$

Corollary 2. *If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vertices of a spherical equilateral triangle with lateral (spherical) length α, i.e. $\mathbf{v}_i$s are vectors on the standard unit sphere S^{m-1} and $\mathbf{v}_i \cdot \mathbf{v}_j = \cos\alpha$ for $i \neq j$, then*

$$3\cos^2\alpha \leq 1 + 2\cos^3\alpha$$

Corollary 3. *Let $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ be a regular spherical n-simplex on S^m. If $\mathbf{v}_i \cdot \mathbf{v}_j = \cos\alpha$ for $i \neq j$, then*

$$(n\cos\alpha + 1)(1 - \cos\alpha)^n \geq 0 \tag{6}$$

Proof. It follows from $B_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \cos\alpha$ for $i \neq j$ and $B_{ii} = 1$ that $det(B) = (n\cos\alpha + 1)(1 - \cos\alpha)^n$.

It follows from the corollary that the spherical length α of the edges of regular n-simplex on the sphere S^n is bounded by $\cos^{-1}(-1/n)$, which converges to $\pi/2$ as n goes to infinity.

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