International Journal of Pure and Applied Mathematics
Volume 89 No. 4 2013, 449-459
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i4.2
ijpam.eu

THE HANKEL TRANSFORM OF $\delta^{(\nu)}(x-a)$ AND DISTRIBUTIONAL PRODUCT OF $\delta^{(m)}(x-a).\delta^{(l)}(x-b)$

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Abstract: One of the problem in distributions theory is the lack of definitions for products and power of distributions in general. In Physics (c.f. [5], p. 141), oneself finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. Chankuan Li ([4]) derives that $\delta^2(x)=0$ on even-dimension space by applying the Laurent expansion of $r=i$. Koh and Li in ([6]) give a sense to distribution δ^k and $(\delta')^k$ for some k, using the concept of neutrix limit. In this paper, using the Hankel Transform of Generalized function of $\delta^{(m)}(x-a)$, we give a sense to distributional product of

$$\delta^{(m)}(x-a).\delta^{(l)}(x-b)$$

where m and l are non-negative integers, For the case $a=b=0$ we obtain the distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$, in particular we obtain the product $\delta(x).\delta(x)$. In the other hand, using the distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$ we obtain a formula to power of distributions $\left(\delta^{(m)}(x)\right)^h$ where m and h are non-negative integers.

Key Words: distributional product, Dirac delta, propieties of distributions

1. Introduction

One of the problems in distribution theory is the lack of definitions for products

Received: May 28, 2013

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and power of distributions in general. In Physics (c.f. [5], p. 141), oneself finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. Koh and Li in ([6]) give a sense to distribution δ^k and $(\delta')^k$ for some k, using the concept of neutrix limit. In this paper, using the Hankel Transform of Generalizaed function of $\delta^{(v)}(x)$ we give a sense to distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$, in particular we obtain the product $\delta(x).\delta(x)$. On the other hand, as consequence of our product we obtain a formula to power of distributions $\left(\delta^{(v)}(x)\right)^s$ where v and s are non-negative integers.

Let $U(t) \in S'_{R^+}$ where S'_{R^+} is the dual of S_{R^+}. S'_{R^+} is the space of function $f \in S$ defined in $R^+ = \{t : t > 0\}$. The Hankel Transform of $U(t)$ will be, by

$$\langle H\{U(t), \varphi(s)\}\rangle = \langle U(t), H\{\varphi(s)\}\rangle \tag{1.1}$$

for every $\varphi \in S_{R^+}$ ([1]).

By Hankel Transform of the function $f(x)$ of order $\mu \geq -\frac{1}{2}$, we mean the function $g(s)$, $0 < s < \infty$ defined by the following formula

$$g(s) = H_\mu\{f(t)\} = \int_0^\infty f(t)\sqrt{ts}J_\mu(ts)\,dt \tag{1.2}$$

([2], page 127), where $J_\mu(z)$ is the Bessel function of first kind and other μ defined by

$$J_\mu(z) = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\mu+2k}}{k!\Gamma(\mu+k+1)} \tag{1.3}$$

and $\Gamma(z)$ is the function gamma defined by

$$\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx.$$

Using the definition (1.1) and (1.2) we obtain the Hankel Transform of generalized function

$$\delta^{(v)}(x-a)$$

of order $\mu \geq -\frac{1}{2}$, where

$$\left\langle \delta^{(v)}(x-a), \varphi(x)\right\rangle = (-1)^v \varphi^{(v)}(a) \tag{1.4}$$

for $\varphi \in G_\mu$ ([2], page 144), a is a real non-negative number, v a non-negative integer abd G_μ is a testing function space as defined in ([2], pages 38-41, 129, 130 and 131).

In fact, form (1.1) and (1.2) we have

$$\begin{aligned}\left\langle H_\mu\left\{\delta^{(v)}(x-a)\right\},\varphi(s)\right\rangle &= \left\langle \delta^{(v)}(x-a), H_\mu\{\varphi(s)\}\right\rangle = \\ &= \left\langle \delta^{(v)}(x-a), \int_0^\infty \varphi(s)\sqrt{xs}J_\mu(xs)\,ds\right\rangle = \\ &= (-1)^v\left\{\frac{d^v}{dx^v}\left(\int_0^\infty \varphi(s)\sqrt{xs}J_\mu(xs)\,ds\right)\right\}\Bigg|_{x=a} = \\ &= \left\langle (-1)^v\frac{d^v}{dx^v}\sqrt{xs}J_\mu(xs)\Bigg|_{x=a}, \varphi(s)\right\rangle. \qquad (1.5)\end{aligned}$$

Therefore,

$$H_\mu\left\{\delta^{(v)}(x-a)\right\} = \left\{(-1)^v\left(\frac{d^v}{dx^v}\sqrt{xs}J_\mu(xs)\right)\right\}\Bigg|_{x=a} \qquad (1.6)$$

The formula (1.6), appear in ([2], page 144).

On the other hand, using (1.3) and taking into account that $0<s<\infty$ we have

$$\begin{aligned}&\left\{\left(\tfrac{d^v}{dx^v}\sqrt{xs}J_\mu(xs)\right)\right\}\Big|_{x=a} = \\ &= \left(\mu+\tfrac{1}{2}\right)\left(\mu+\tfrac{1}{2}-1\right)\cdots\left(\mu+\tfrac{1}{2}-(v-1)\right)a^{\mu+\frac{1}{2}-v}s^{\mu+\frac{1}{2}}+ \qquad (1.7)\\ &+\textstyle\sum_{k\geq1}\frac{(-1)^k\left(\mu+2k+\frac{1}{2}\right)\cdots\left(\mu+2k+\frac{1}{2}-(v-1)\right)}{k!2^{\mu+2k}\Gamma(\mu+k+1)}a^{\mu+2k+\frac{1}{2}-v}s^{\mu+2k+\frac{1}{2}}\end{aligned}$$

where $J_\mu(z)$ is defined by(1.3) and

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \qquad (1.8)$$

([3], page 48).

Now, if $\mu = v-\frac{1}{2}$, from (1.6) and using (1.7) we have

$$\begin{aligned}H_{v-\frac{1}{2}}\left\{\delta^{(v)}(x-a)\right\} &= \frac{v!(-1)^v}{2^{v-\frac{1}{2}}\Gamma\left(v+\frac{1}{2}\right)}s^v + \\ &+(-1)^v\sum_{k\geq1}\frac{(-1)^k(2k+1)(2k+2)\cdots(v+2k-1)(v+2k)}{k!2^{2k+v-\frac{1}{2}}\Gamma\left(v-\frac{1}{2}+k+1\right)}a^{2k}s^{v+2k} \qquad (1.9)\end{aligned}$$

Now, putting $a = 0$ in (1.9) and considering that

$$s^{\nu} = s^{\nu}_{+} + (-1)^{\nu} s^{\nu}_{-},$$

where

$$s^{\nu}_{+} = \begin{cases} s^{\nu} & \text{if } s \geq 0 \\ & \text{and} \\ 0 & \text{if } s < 0 \end{cases} \qquad ([3], \text{page47}) \tag{1.10}$$

and

$$s^{\nu}_{-} = \begin{cases} \mid s \mid^{\nu} & \text{if } s < 0 \\ & \text{and} \\ 0 & \text{if } s \geq 0 \end{cases} \qquad ([3]),\text{page49}),$$

we obtain the following formula

$$H_{v-\frac{1}{2}} \left\{ \delta^{(v)} (x) \right\} = \frac{(-1)^{v} \, v!}{2^{v-\frac{1}{2}} \Gamma \left(v + \frac{1}{2} \right)} s^{v}_{+} \tag{1.11}$$

where $v = 0, 1, 2, \ldots$

From (1.9) and using the formula

$$\Gamma (z + h) = z.(z + 1) \ldots (z + h - 1) \Gamma (z)$$

we have,

$$\mathcal{H}_{v-\frac{1}{2}} \left\{ \delta^{(v)} (x - a) \right\} = \sum_{i \geq 0} \frac{(-1)^{v} (-1)^{i} \Gamma (2i + 1 + v)}{i! 2^{2i+v-\frac{1}{2}} \Gamma \left(v - \frac{1}{2} + i + 1 \right)} a^{2i} s^{v+2i}_{+} \tag{1.12}$$

([?], page 3, formula 9).

By putting $a = 1$ in (1.12) we obtain the formula

$$\mathcal{H}_{v-\frac{1}{2}} \left\{ \delta^{(v)} (x - 1) \right\} = \sum_{i \geq 0} \frac{(-1)^{v} (-1)^{i} \Gamma (2i + 1 + v)}{i! 2^{2i+v-\frac{1}{2}} \Gamma \left(v - \frac{1}{2} + i + 1 \right)} s^{v+2i}_{+} \tag{1.13}$$

From (1.11) we have

$$\delta^{(v)} (x) = \frac{(-1)^{v} \, v!}{2^{v-\frac{1}{2}} \Gamma \left(v + \frac{1}{2} \right)} \mathcal{H}^{-1}_{v-\frac{1}{2}} \left\{ s^{v}_{+} \right\} \tag{1.14}$$

where for $\mu \geq \frac{1}{2}$ the Hankel transform $\mathcal{H}^{-1}_{\mu}$ defined by the same formula as is direct Hankel Transform $\mathcal{H}_{\mu}$ ([2], p. 127).

Using the formula

$$t_{+}^{\lambda-1} * t_{+}^{\mu-1} = B(\lambda,\mu)\, t_{+}^{\lambda+\mu-1} \tag{1.15}$$

for λ, μ complex numbers such that $\operatorname{Re}(\lambda) > 1$, $\operatorname{Re}(\mu) > 1$ and $\operatorname{Re}(\lambda+\mu) > 1$ when

$$B(\lambda,\mu) = \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)} \tag{1.16}$$

we have,

$$s_{+}^{m} * s_{+}^{l} = \frac{\Gamma(m+1)\,\Gamma(l+1)}{\Gamma(m+l+2)} s_{+}^{l+m+1} \tag{1.17}$$

for $m, l = 0, 1, 2, \ldots$.

For (1.12) and (1.17) we have,

$$\begin{aligned}
&\mathcal{H}_{\mu-\frac{1}{2}}\left\{\delta^{(m)}(x-a)\right\} * H_{k-\frac{1}{2}}\left\{\delta^{(l)}(x-a)\right\} = \\
=\ &\left(\sum_{i\geq 0} \frac{(-1)^{m}(-1)^{i}\,\Gamma(2i+1+m)}{i!2^{2i+\mu-\frac{1}{2}}\Gamma\left(\mu-\frac{1}{2}+i+1\right)} a^{2i}\right) \cdot \\
&\cdot\left(\sum_{j\geq 0} \frac{(-1)^{l}(-1)^{j}\,\Gamma(2j+1+l)}{j!2^{2j+k-\frac{1}{2}}\Gamma\left(k-\frac{1}{2}+j+1\right)} a^{2j}\right)\left(s_{+}^{m+2i} * s_{+}^{k+2j}\right) \\
=\ &\sum_{i\geq 0}\sum_{j\geq 0} \frac{(-1)^{m}(-1)^{i}(-1)^{l}(-1)^{j}\,\Gamma(2i+m+1)\,\Gamma(2j+l+1)}{i!j!2^{2i+\mu-\frac{1}{2}+2j+k-\frac{1}{2}}\Gamma\left(\mu-\frac{1}{2}+i+1\right)\Gamma\left(k-\frac{1}{2}+j+1\right)} \cdot \\
&\cdot \frac{\Gamma(m+2i)\,\Gamma(l+2j)}{\Gamma(\mu+2i+k+2j)} s_{+}^{m+2i+k+2j}
\end{aligned} \tag{1.18}$$

Taking into account the formula (1.14) and the propertie (1.18) we defined the product of $\delta^{(u)}(x-a)\,.\delta^{(v)}(x-b)$ through the following form:

Definition 1. Let m, l and r non-negative integers and $\delta^{(v)}(x-a)$ the distribution defined by the equation (1.4) and taking into account the formula (1.18) we defined the multiplicative product of $\delta^{(m)}(x-a)\,.\delta^{(l)}(x-b)$ by mean of the following formula

$$\begin{aligned}
&\delta^{(m)}(x-a)\,.\delta^{(l)}(x-b) = \\
=\ &\mathcal{H}^{-1}_{m+l+r+1-\frac{1}{2}}\left\{\mathcal{H}_{m-\frac{1}{2}}\left\{\delta^{(m)}(x-a)\right\} * \mathcal{H}_{l-\frac{1}{2}}\left\{\delta^{(l)}(x-b)\right\}\right\}
\end{aligned} \tag{1.19}$$

where $\mathcal{H}^{-1}$ is the inverse Hankel Transform ([2], page 27).

Now using the definition (1.19) and taking into account the formula (1.13) we arrive at the following formula

$$\delta^{(m)}(x-a).\delta^{(l)}(x-b) =$$

$$\mathcal{H}^{-1}_{m+l+r}\left\{\sum_{r\geq 0}\sum_{j\geq 0}\frac{(-1)^{m+l}(-1)^{r}\Gamma(2(r-j)+m+1)\Gamma(m+2(r-j)+1)\Gamma(l+2j+1)}{(-j)!j!2^{2r+m+l-1}\Gamma\left(m-\frac{1}{2}+r-j+1\right)\Gamma\left(l-\frac{1}{2}+j+1\right)\Gamma(m+l+2r+2)}\right.$$

$$\left. a^{2(r-j)}.b^{2j}\frac{\Gamma(m+2(r-j)+1)\Gamma(l+2j+1)}{\Gamma(m+l+2r+2)}s_{+}^{m+l+r+1}\right\}. \tag{1.20}$$

Taking into account that

$$\mathcal{H}^{-1}_{m+l+r+1-\frac{1}{2}}\left\{s_{+}^{m+r+l+1}\right\} = \frac{2^{l+r+1-\frac{1}{2+m}}\Gamma\left(m+l+r+1+\frac{1}{2}\right)}{(-1)^{m+l+r+1}(m+l+r+1)!}\delta^{(m+l+r+1)}(x) \tag{1.21}$$

from (1.20) we obtain the formula

$$\delta^{(m)}(x-a).\delta^{(l)}(x-b) =$$

$$\sum_{r\geq 0}\sum_{j=0}^{r}\frac{(-1)2^{\frac{3}{2}}\Gamma(2(r-j)+m+1)\Gamma(2j+l+1)\Gamma(m+2(r-j)+1)\Gamma(l+2j+1)}{2^{r}(r-j)j!\Gamma\left(m+r-j+1-\frac{1}{2}\right)\Gamma\left(l+j+1-\frac{1}{2}\right)}.$$

$$.\frac{\Gamma\left(m+r+l+1+\frac{1}{2}\right)a^{2(r-j)}b^{2j}}{\Gamma(m+l+2r+2)(m+l+r+1)!}\delta^{(m+l+r+1)}(x) \tag{1.22}$$

If $a=b$ from (1.22) we have

$$\delta^{(m)}(x-a).\delta^{(l)}(x-a) =$$

$$\sum_{r\geq 0}\sum_{j=0}^{r}\frac{(-1)2^{\frac{3}{2}}\Gamma(2(r-j)+m+1)\Gamma(2j+l+1)\Gamma(m+2(r-j)+1)\Gamma(l+2j+1)}{2^{r}(r-j)j!\Gamma\left(m+r-j+1-\frac{1}{2}\right)\Gamma\left(l+j+1-\frac{1}{2}\right)} \tag{1.23}$$

$$.\frac{\Gamma\left(m+r+l+1+\frac{1}{2}\right)a^{2r}}{\Gamma(m+l+2r+2)(m+l+r+1)!}\delta^{(m+l+r+1)}$$

If $a=b=0$ from (1.22) we have

$$\delta^{(m)}(x).\delta^{(l)}(x) =$$

$$= \sum_{r\geq 0}\frac{(-1)2^{\frac{1}{2}}\Gamma(m+1)\Gamma(l+1)\Gamma(m+1)\Gamma(l+1)\Gamma\left(m+l+1+\frac{1}{2}\right)}{2^{r}\Gamma\left(m+1-\frac{1}{2}\right)\Gamma\left(l+1-\frac{1}{2}\right)\Gamma(m+l+2)(m+l+1)!}\delta^{(m+l+1)}$$

$$= \frac{(-1)\,2^{\frac{1}{2}}\,(m!l!)^2}{((m+l+1)!)^2}\frac{\Gamma\left(m+l+1+\frac{1}{2}\right)}{\Gamma\left(m+1-\frac{1}{2}\right)\Gamma\left(l+1-\frac{1}{2}\right)}\delta^{(m+l+1)}(x) \qquad (1.24)$$

By putting $m=l=0$ in(1.24) we obtain the following formula

$$\delta(x).\delta(x) = c\delta'(x) \qquad (1.25)$$

where

$$\begin{aligned} c &= \frac{(-1)\,2\sqrt{2}\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} = \\ &= \frac{(-1)\,2\sqrt{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)} = \\ &= \frac{(-1)\sqrt{2}}{\sqrt{\pi}} = \\ &= (-1)\sqrt{\frac{2}{\pi}} \end{aligned} \qquad (1.26)$$

2. The Distributions $\left(\delta^{(m)}(x-a)\right)^2$ and $\left(\delta^{(m)}(x)\right)^s$

From (1.22) we have,

$$\delta^{(m)}(x-a).\delta^{(l)}(x-a) = \sum_{r\geq 0} A_{m,l,r,j}a^{2r}\delta^{(m+l+r+1)}(x) \qquad (2.1)$$

where

$$A_{m,l,r,j} = \sum_{j=0}^{r}\left(\frac{(-1)2^{\frac{3}{2}}\Gamma(2(r-j)+m+1)\Gamma(2j+l+1)}{2^r(r-j)!j!\Gamma\left(m+r-j+1-\frac{1}{2}\right)}\right. \qquad (2.2)$$
$$\left.\frac{\Gamma(m+2(r-j)+1)\Gamma(l+2j+1)\Gamma\left(m+r+l+1+\frac{1}{2}\right)}{\Gamma\left(l+j+1-\frac{1}{2}\right)\Gamma(m+l+2r+2)\Gamma(m+l+r+2)}\right)$$

From (2.1) we have,

$$\delta^{(m)}(x-a).\delta^{(m)}(x-a) = \sum_{r\geq 0} B_{m,m,r,j}a^{2r}\delta^{(2m+r+1)}(x) \qquad (2.3)$$

where

$$B_{m,m,r,j} = A_{m,m,r,j} \qquad (2.4)$$

and $A_{m,m,r,j}$ is defined by (2.2).

From (2.3) we give sense to distribution $\left(\delta^{(m)}(x-a)\right)^2$ by meaning the following definition:

Definition 2. The distribution $\left(\delta^{(m)}(x-a)\right)^2$ is by definition

$$\left(\delta^{(m)}(x-a)\right)^2 = \delta^{(m)}(x-a).\delta^{(m)}(x-a) \tag{2.5}$$

Therefore, by using the formula (2.3) and (2.5) we have

$$\left(\delta^{(m)}(x-a)\right)^2 = \sum_{r\geq 0} B_{m,m,r,j} a^{2r} \delta^{(2m+r+1)}(x) \tag{2.6}$$

In particular by putting $a=0$ in (2.6) we have,

$$\left(\delta^{(m)}(x)\right)^2 = C_{m,m}\delta^{(2m+1)}(x) \tag{2.7}$$

where

$$C_{m,m} = (-1)\, 2^{\frac{3}{2}} \left(\frac{m!}{(2m+1)!}\right)^2 \frac{\Gamma\left(2m+\frac{3}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)} \tag{2.8}$$

From (2.7) and (1.24) we have,

$$\left(\delta^{(m)}(x)\right)^3 = C_{m,m}.C_{m,2m+1}\delta^{(3m+2)}(x) \tag{2.9}$$

By interaction h-times the above formulae, we arrive to

$$\left(\delta^{(m)}(x)\right)^h = C_{m,m}.C_{m,2m+1}.\dots.C_{m,(h-1)m+h-2}\delta^{(hm+h-1)} \tag{2.10}$$

where

$$C_{m,m}.C_{m,2m+1}.\dots.C_{m,(h-1)m+h-2} = D_{m,(h-1)m+h-2} =$$

$$= (-1)\, 2^{\frac{3}{2}} \left(\frac{m!}{(2m+1)!}\right)^2 \frac{\Gamma\left(2m+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}.\, (-1)\, 2^{\frac{3}{2}} \left(\frac{m!(2m+1)!}{(3m+2)!}\right)^2 \frac{\Gamma\left(3m+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(2m+1+\frac{1}{2}\right)}.$$

$$.\,(-1)\, 2^{\frac{3}{2}} \left(\frac{m!(3m+2)!}{(4m+3)!}\right)^2 \frac{\Gamma\left(4m+2+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(3m+2+\frac{1}{2}\right)} \cdots$$

$$\cdots (-1)\, 2^{\frac{3}{2}} \left(\frac{m!((h-1)m+h-2)!}{(hm+h-1)!}\right)^2 \frac{\Gamma\left(hm+h-2+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left((h-1)m+h-2+\frac{1}{2}\right)} =$$

$$= \frac{\left((-1)2^{\frac{3}{2}}\right)^{h-2}\left((m!)^2\right)^{h-2}((2m+1)!)^2((3m+2)!)^2\cdots(((h-1)m+m+h-2)!)^2}{((3m+2)!)^2((4m+3)!)^2\cdots((hm+h-1)!)^2}$$

$$.\frac{\Gamma\left(3m+1+\frac{1}{2}\right)\Gamma\left(4m+2+1+\frac{1}{2}\right)\cdots\Gamma\left(hm+h-2+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(2m+1+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(3m+2+\frac{1}{2}\right)\cdots\Gamma\left(m+\frac{1}{2}\right)\Gamma\left((h-1)m+h-2+\frac{1}{2}\right)} =$$

$$= \frac{\left((-1)2^{\frac{3}{2}}\right)^{h-1}(m!)^2(m!)^2(m!)^2\dots(m!)^2((2m+1)!)^2((3m+2)!)^2((4m+3)!)^2\cdots(((h-1)m+h-2)!)^2}{((2m+1)!)^2((3m+2)!)^2((4m+3)!)^2\cdots(((h-1)m+h-2)!)^2((hm+h-1)!)^2}$$

$$.\frac{\Gamma\left(2m+1+\frac{1}{2}\right).\Gamma\left(3m+1+1+\frac{1}{2}\right)\Gamma\left(4m+2+1+\frac{1}{2}\right)\cdots\Gamma\left(hm+h-2+1+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(2m+1+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(3m+2+\frac{1}{2}\right)\cdots\Gamma\left(m+\frac{1}{2}\Gamma\left((h-1)m+h-2+\frac{1}{2}\right)\right)} =$$

$$= \frac{\left((-1)2^{\frac{3}{2}}\right)^{h-1}\left((m!)^2\right)^{h-1}\Gamma\left(hm+h-2+1+\frac{1}{2}\right)}{((hm+h-1)!)^2\left(\Gamma\left(m+\frac{1}{2}\right)\right)^h}. \tag{2.11}$$

Definition 3. Therefore, form (2.10) and using (2.11) we obtain a iterative formula of $\left(\delta^{(m)}(x)\right)^h$:

$$\left(\delta^{(m)}(x)\right)^h = E_{m,h}.\delta^{(hm+h-1)}(x) \tag{2.12}$$

where

$$E_{m,h} = \frac{\left((-1)\, 2^{\frac{3}{2}}\right)^{h-1}\left((m!)^2\right)^{h-1}\Gamma\left(hm+h-2+1+\frac{1}{2}\right)}{((hm+h-1)!)^2\left(\Gamma\left(m+\frac{1}{2}\right)\right)^h} \tag{2.13}$$

and $h = 2, 3, 4, \dots$.

In particular, by putting $m=0$ in (2.12) and (2.13) we obtain the following formula

$$\begin{aligned}(\delta(x))^h &= E_{0,h}\delta^{(h-1)}(x) = \\ &= \frac{\left((-1)\,2^{\frac{3}{2}}\right)^{h-1}\Gamma\left(h-1+\frac{1}{2}\right)}{((h-1)!)^2\left(\Gamma\left(m+\frac{1}{2}\right)\right)^h}\delta^{(h-1)}(x) \qquad (2.14)\end{aligned}$$

for $h=2,3,4,\ldots$.

Simialrly, by putting $m=1$ in (2.13) we obtain the following formula

$$\left(\delta'(x)\right)^h = E_{1,h}\delta^{(2h-1)}(x) \qquad (2.15)$$

where

$$E_{1,h} = \frac{\left((-1)\,2^{\frac{3}{2}}\right)^{h-1}\Gamma\left(2h-1+\frac{1}{2}\right)}{((2h-1)!)^2\left(\Gamma\left(\frac{3}{2}\right)\right)^h}. \qquad (2.16)$$

and $h=2,3,4,\ldots$.

Acknowledgments

This work was partially supported by Comisión de Investigaciones Científicas de la provincia de Buenos Aires (C.I.C.), Argentina.

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