International Journal of Pure and Applied Mathematics
Volume 90 No. 1 2014, 101-107
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v90i1.12

ijpam.eu

FIRST NOTE ON THE SHAPE OF *S*-CONVEXITY

I.M.R. Pinheiro
P.O. Box 12396, A'Beckett St, Melbourne
Victoria, 8006, AUSTRALIA

Abstract: In this note, we present a few important scientific remarks regarding the shape of $S-$convexity.

AMS Subject Classification: 26A51
Key Words: analysis, convexity, definition, s-convexity, geometry, shape, $S-$convexity, $s-$convex function, $S-$convex function

1. Introduction

We seem to have progressed quite a lot in terms of the analytical definitions for the real $S-$convex functions (see, for instance, Minima Domain Intervals and the $S-$convexity, as well as the Convexity, Phenomenon, paper that comes straight after [5] in our series aiming at improving the wording of the definition of the $S-$convexity phenomenon). One of the resulting analytical definitions in ([6]) is:

Definition 2. *A function $f : X- > \Re$, where $|f(x)| = f(x)$, is told to belong to K_s^2 if the inequality*

$$f(\lambda x + (1-\lambda)(x+\delta))$$
$$\leq \lambda^s f(x) + (1-\lambda)^s f(x+\delta)$$

holds $\forall\lambda/\lambda \in [0,1]$; $\forall x/x \in X$; $s = s_2/0 < s_2 \leq 1$;$X/X \subseteq \Re_+ \wedge X = [a,b]$; $\forall\delta/0 < \delta \leq (b-x)$.

Received: December 14, 2013

Remark 1. If the inequality is obeyed in the reverse[1] situation by f, then f is told to be s_2-concave.

We now need to improve the wording of the geometric definition for $S-$convex functions as well.

The geometric definition of convexity has been worded by us in our work on minima domain intervals in the following way:

Definition 3. *A real function $f : X- > Y$ is called convex if and only if, for all choices $(x_1; y_1)$ and $(x_2; y_2)$, where $\{x_1, x_2\} \subset X$, $\{y_1, y_2\} \subset Y$, $Y = Im_f$, and $x_1 \neq x_2$[2], it happens that the chord drawn between $(x_1; y_1)$ and $(x_2; y_2)$ does not contain any point with height, measured against the vertical Cartesian axis, that be inferior to the height of its horizontal equivalent in the curve representing the ordered pairs of f in the interval considered for the chord in terms of distance from the origin of the Cartesian axis.*

We would like to word the geometric definition of the $S-$convex functions in a similar way:

Definition 4. *A real function $f : X- > Y$, for which $|f(x)| = f(x)$, is called $S-$convex if and only if, for all choices $(x_1; y_1)$ and $(x_2; y_2)$, where $\{x_1, x_2\} \subset X$, $\{y_1, y_2\} \subset Y$, $Y = Im_f$, and $x_1 \neq x_2$, it happens that the line drawn between $(x_1; y_1)$ and $(x_2; y_2)$ by means of the expression $(1-\lambda)^s y_1 + \lambda^s y_2$, where $\lambda \in [0, 1]$, does not contain any point with height, measured against the vertical Cartesian axis, that be inferior to the height of its horizontal equivalent in the curve representing the ordered pairs of f in the interval considered for the line in terms of distance from the origin of the Cartesian axis.*

To prove that the geometric rule for pertinence to the $S-$convex class of functions is the one we present, we remind the reader that $(1 - \lambda)^s y_1 + \lambda^s y_2 \geq (1 - \lambda)y_1 + \lambda y_2$ due to the allowed (by the definition) values for λ, s, y_1, and y_2.

We prove that the geometric limiting line for $S-$convexity is continuous in Section 2.

In Section 3, we prove that, as s decreases in value (as the distance from the convexity-limiting line is increased), the length of the limiting line increases, therefore we prove that we have more functions in the 1/4-convex class than in the 1/2-convex class, for instance, what provides us with certainty that $S-$convexity is a proper extension of convexity, geometrically speaking.

---

[1] Reverse here means $>$, not $\geq$.

[2] y_1 can be equal to y_2, for the constant function, for instance, is convex. Notice that we are also fixing the wording of the geometric definition of convex function here.

5. Continuity

We felt the need of proving that the line $(1-\lambda)^s y_1 + \lambda^s y_2$ is continuous due to the questioning that we have been subjected to by really important researchers in Mathematics. Basically, a person could think that this line would break because it would be a lift on the line that used to be straight and used to occupy a space of size x, let's say. This line would have to then stretch to cover the same distance in a 'bent trajectory', rather than in a straight trajectory.

We are using the same basic fractures that we have used with convexity, for we have $a+b=1$ in the same way, so that there is a line of reasoning that may make us doubt that making that lift, a^s, b^s, would not imply breaking the limiting line somehow.

The lift would make 0.25 become 0.5, for instance, when $s = 0.5$.

To prove that the line does not break, all we have to do is using a few theorems from Real Analysis.

We know, for instance, that both the sum and the product of two continuous functions are continuous functions (see, for instance, [3]. Notice that λ^s is continuous, given that $0 \leq \lambda \leq 1$. Also y_1 and y_2 will be constants, therefore could be seen as constant functions, which are continuous functions. $(1-\lambda)$ will not be negative, what will make $(1-\lambda)^s$ be continuous. As a consequence, $(1-\lambda)^s y_1 + \lambda^s y_2$ will be continuous.

Notice that $f(\lambda) = (1-\lambda)^s y_1 + \lambda^s y_2$ is C^∞, that is, is smooth (see [1], for instance).

We do notice that we will have problems, for instance, in the first derivative of $f(\lambda)$ of the sort 0^0 (see [7], for instance), when $\lambda \in \{0, 1\}$ and $s = 1$, but, in excluding $s = 1$ from our set of possible values exclusively for derivative matters, what we can easily do, given that the expression of $f(\lambda)$ will be very different and much simpler when $s = 1$, we disappear with those problems. We then say that $s \in (0, 1)$ when $f'(\lambda) = -s(1-\lambda)^{s-1} y_1 + s\lambda^{s-1} y_2$ and $s = 1$ when $f'(\lambda) = y_2 - y_1$.

Because the coefficients that form the convexity limiting line use 100% split between the addends and form straight lines and the coefficients that form the $S-$convexity limiting line use more than 100% or 100% split between addends, trivially, given that $(1-\lambda)^s \geq (1-\lambda)$ and $\lambda^s \geq \lambda$, we know that the limiting line for $S-$convexity lies always above or over the limiting line for convexity, and contains two points that always belong to both the convexity and the $S-$convexity limiting lines (first and last or $(x_1; y_1)$ and $(x_2; y_2)$).

We now have then proved, in a definite manner, also in this paper, that our limiting line, for the $S-$convexity phenomenon, is smooth, continuous, and located above or over the limiting line for the convexity phenomenon. Our $S-$convexity limiting line should also be concave when 'seen' from the limiting convexity line for the same points ($(x_1; y_1)$ and $(x_2; y_2)$) (taking away the cases in which $y_1 = y_2 = 0$ or $s = 1$), as we asserted to be the case in our talks at the Victoria University of Technology (2001) and at the Adelaide University (2005).

6. Arc Length

Arc length is defined as the length along a curve,

$$s \equiv \int_\gamma |dl|,$$

where dl is a differential displacement vector along a curve γ (see [2]).

In Cartesian coordinates, that means that the Arc Length of a curve is given by

$$p \equiv \int_a^b \sqrt{1 + f'^2(x)}dx$$

whenever the curve is written in the shape $r(x) = x\hat{\boldsymbol{x}} + f(x)\hat{\boldsymbol{y}}$.

Our limiting curve for $S-$convexity could be expressed as a function of λ in the following way:

$$f(\lambda) = (1-\lambda)^s y_1 + \lambda^s y_2.$$

In deriving the above function in terms of λ, we get:

$$f'(\lambda) = -s(1-\lambda)^{s-1}y_1 + s\lambda^{s-1}y_2.$$

With this, our arc length formula will return:

$$p \equiv \int_0^1 \sqrt{1 + [-s(1-\lambda)^{s-1}y_1 + s\lambda^{s-1}y_2]^2}d\lambda.$$

We will make use of a constant function, and we know that every constant function is convex, therefore also S-convex, for any s, to study the limiting line for $S-$convexity better.

We choose $f(x) = 1.2$ to work with.

We then have:

$$p \equiv \int_0^1 \sqrt{1 + 1.44s^2[\lambda^{s-1} - (1-\lambda)^{s-1}]^2}d\lambda.$$

Notice that if $\lambda \notin \{0,1\}$, $s \to 0 \Rightarrow p \to 1$ and $s \to 1 \Rightarrow p \to 1$. Because the lengths are equal, we have the intuition that the line does not change sizes as s decreases in value. However, $s = 0$ implies that we have a simple sum (take away the case in which $\lambda \in \{0,1\}$), not an average of some sort. In fact, we have $y_1 + y_2$ all the time.

Because S-convexity ($S = 1$ means convexity) makes its limiting line from summing percentages of two heights, and percentages that together, without the exponent, give us 100%, it does not make sense considering 100% of each one of the heights, therefore possibly a total of 200% in our mix (could be the case with $s = 0$) almost all the time or all the time. That is a geometric view on what is going on with these definitions, but good analytical reasons to never consider $s = 0$ are not missing. Notice that $s = 0$ gives us a limiting line that is almost certain to not contain y_1 or y_2 ($f(\lambda) = y_1 + y_2$, our limiting line when $s = 0$, can only contain y_1 if $y_2 = 0$ and can only contain y_2 if $y_1 = 0$ if we assume that $y_1 \neq y_2$).

Also notice that if we exclude 0 from our set of possible values for s, then we recover 'sanity', that is, the length of the limiting line will finally grow as we distance ourselves from the convexity limiting line through increasing the value of our percentages in the partitions.

Notice that 0.25 will become 0.5 when raised to 0.5 and its supplement through the formula $(1 - \lambda)$, 0.75, will become approximately 0.87.

In convexity, our results would have been 0.25 and 0.75 instead, that is, 100% and 16% extra is gotten with $S-$convexity, respectively.

We finish this section with a table[3] containing three of the possible values for s and their respective arc lengths (good approximations, apart from the first value, which is precise) when $f(x) = 1.2$:

s	Arc Length
1	1
0.5	1.57
0.25	2

Because of the table, we notice that the naive formula from [4] gives us an excellent approximation to the arc lengths for the S-convexity limiting line, so that there is a good chance that we can always use the naive formula to calculate the arc length in this situation.

[3] The first value for Arc Length in the table has been attained through simple substitution in the formula. The second value has been attained through using the formula for circumference length. The third value has been attained through the naive formula from [4]. Hand measurement has returned 2 as a result for the third value.

7. Maximum Height

The maximum height of the $S-$convexity limiting curve is reached when $\lambda = 0.5$ if f is constant and $|f(x)| = f(x)$ because the first derivative of the function describing the limiting line gives us zero for $\lambda = 0.5$ and changes sign from positive to negative there.

8. Conclusion

In this paper, we have presented our first proposal of geometric definition for the phenomenon $S-$convexity together with a revised proposal of geometric definition for the phenomenon Convexity. To explain our proposal, alternative argumentation as for why the shape of $S-$convexity is what we have declared it to be in our talk in 2001 at the Victoria University of Technology, talk given to the members of the Research Group on Inequalities and Applications who were working there on a regular basis, and in our talk in 2005 at the University of Adelaide, talk given to diverse audience, has been introduced.

We have managed to also present a conjecture of major importance that connects the s in $S-$convexity with the arc length function, giving us a relationship that determines how bent the limiting curve is in function of s. The conjecture is formed from the analysis of a few experimental results.

Future work of ours will bring more on the geometry of $S-$convexity, including some theory regarding the remaining group of $S-$convex real functions, that for which $|f(x)| = -f(x)$.

References

[1] Planetmath editors, *Continuously Differentiable*, http://planetmath.org/encyclopedia/ContinuouslyDifferentiable.html, Accessed on the 9-th of December of 2011.

[2] Wolfram Web Editors, ArcLength, http://mathworld.worlfram.com/ArcLength.html, Accessed on the 5-th of December of 2010.

[3] [Hus95] L.S. Hush, *Continuous Functions*, http://archives.math.utk.edu/visual.calculus/1/continuou s.5/index.html (1995), Accessed on the 8-th of December of 2011.

[4] G.P. Michon, *Perimeter of an Ellipse*,
http://www.numericana.com/answer/ellipse.htm (2010), Accessed on the 12-th of December of 2011.

[5] M.R. Pinheiro, First note on the definition of s2-convexity, *Advances in Pure Mathematics*, **1** (2011). 12.

[6] M.R. Pinheiro, Minima domain intervals and the S-convexity, as well as the convexity phenomenon, *Advances in Pure Mathematics*, **3**, No. 1 (2013).

[7] E.W. Weisstein, *Indeterminate*,
http://mathworld.worlfram.com/Indeterminate.html (2010), Accessed on the 12-th of December of 2010.