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A NON-UNIFORM BOUND ON POISSON APPROXIMATION FOR RANDOM SUMS OF GEOMETRIC RANDOM VARIABLES

K. Teerapabolarn
Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

Abstract: We determine a non-uniform bound for the distance between the distribution function of random sums of independent geometric random variables and an appropriate Poisson distribution function. Two examples have been given to illustrate the result obtained.

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1. Introduction

Let S_n be a sum $\sum_{i=1}^{n} X_i$ of independently distributed geometric random variables, where $P(X_i = k) = (1-p_i)^k p_i$ for $k = 0, 1, ...$ It is well-known that if all $q_i = (1-p_i)$ are small, the distribution of S_n can be approximated by the Poisson distribution with mean $E(S_n) = \sum_{i=1}^{n} q_i p_i^{-1}$. Correspondingly, the distribution function of S_n can also be approximated by Poisson distribution function with these means. Let $\mathbb{P}\mathbb{S}_n(x_0) = P(S_n \le x_0)$ and $\mathbb{P}_{\lambda_n}(x_0) = \sum_{k=0}^{x_0} \frac{e^{-\lambda_n}\lambda_n^k}{k!}$ be the distribution function of S_n and the Poisson distribution function with mean λ_n

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at $x_0 \in \mathbb{N}$, respectively. In this case, Teerapabolarn [4] used the Stein-Chen method to give a non-uniform bound for the difference of $\mathbb{PS}_n(x_0)$ and $\mathbb{P}_{\lambda_n}(x_0)$ as follows:

$$|\mathbb{PS}_n(x_0) - \mathbb{P}_{\lambda_n}(x_0)| \leq \min\left\{\frac{1-e^{-\lambda_n}}{\lambda_n}, \frac{1}{x_0}\right\}\sum_{i=1}^{n} q_i^2 p_i^{-2}, \tag{1.1}$$

where $x_0 \in \mathbb{N}$. Consider the sum $S_N = \sum_{i=1}^{N} X_i$, where N is a non-negative integer valued random variable and independent of the X_i's. Then S_N is called the *random sums* of independent geometric random random variables. In this study, we are interest to approximate $\mathbb{PS}_N(x_0)$ by $\mathbb{P}_\lambda(x_0)$ when $\lambda = E(\lambda_N)$.

2. Method

Stein's method was originally formulated for normal approximation by Stein [2]. It was adapted and applied to the Poisson case by Chen [1], which is refer to as the Stein-Chen method. Following [3], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{x_0}(x) - \mathbb{P}_\lambda(x_0) = \lambda f_{x_0}(x+1) - x f_{x_0}(x), \tag{2.1}$$

where $x_0, x \in \mathbb{N} \cup \{0\}$, and for $h_{x_0}(x) = 1$ if $x \leq x_0$ and $h_{x_0}(x) = 0$ if $x > x_0$, the solution f_{x_0} is

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^\lambda[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]], & \text{if } x \leq x_0, \\ (x-1)!\lambda^{-x}e^\lambda[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]], & \text{if } x > x_0, \\ 0, & \text{if } x = 0. \end{cases} \tag{2.2}$$

The following lemma gives a non-uniform bound of (2.2).

Lemma 2.1. *For $x_0 \in \mathbb{N}$, let $p_\lambda(x_0+1) = \frac{e^{-\lambda}\lambda^{x_0+1}}{(x_0+1)!}$. Then the following inequality holds:*

$$0 \leq \sup_{x\geq 1} f_{x_0}(x) \leq \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))}{(x_0+1)p_\lambda(x_0+1)} \text{ [3]}. \tag{2.3}$$

3. Result

The following theorem presents non-uniform bounds with different Poisson mean for the distance between $\mathbb{PS}_N(x_0)$ and $\mathbb{P}_\lambda(x_0)$.

Theorem 3.1. *Let $\lambda = E(\lambda_N)$ and $x_0 \in \mathbb{N}$, then we have*

$$|\mathbb{PS}_N(x_0) - \mathbb{P}_\lambda(x_0)| \leq \min\left\{ E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\sum_{i=1}^{N} q_i^2 p_i^{-2}\right), \frac{E\left(\sum_{i=1}^{N} q_i^2 p_i^{-2}\right)}{x_0}\right\} + \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))E|\lambda_N - \lambda|}{(x_0+1)p_\lambda(x_0+1)}, \tag{3.1}$$

where $\mathbb{PS}_N(0) = E(\prod_{i=1}^{N} p_i)$.

Proof. 1. We have

$$\begin{aligned} |\mathbb{PS}_N(x_0) - \mathbb{P}_\lambda(x_0)| &\leq |\mathbb{PS}_N(x_0) - \mathbb{P}_{\lambda_N}(x_0)| + |\mathbb{P}_{\lambda_N}(x_0) - \mathbb{P}_\lambda(x_0)| \\ &= \sum_{n=0}^{\infty} P(N=n)\,|\mathbb{PS}_n(x_0) - \mathbb{P}_{\lambda_n}(x_0)| + |\mathbb{P}_{\lambda_N}(x_0) - \mathbb{P}_\lambda(x_0)| \\ &\leq \sum_{n=0}^{\infty} P(N=n)\min\left\{\frac{1-e^{-\lambda_n}}{\lambda_n}, \frac{1}{x_0}\right\}\sum_{i=1}^{n} q_i^2 p_i^{-2} \\ &\quad + |\mathbb{P}_{\lambda_N}(x_0) - \mathbb{P}_\lambda(x_0)|, \end{aligned} \tag{3.2}$$

where the first term of the right hand side of (3.2) follows from (1.1). Because

$$\begin{aligned} |\mathbb{P}_{\lambda_N}(x_0) - \mathbb{P}_\lambda(x_0)| &= |E\{\lambda f_{x_0}(U_{\lambda_N}+1) - U_{\lambda_N} f_{x_0}(U_{\lambda_N})\}| \\ &= |E\{\lambda E\{[f_{x_0}(U_{\lambda_N}+1)]|\lambda_N\} - E\{[U_{\lambda_N} f_{x_0}(U_{\lambda_N})]|\lambda_N\}\}| \\ &= |E\{\lambda E\{[f_{x_0}(U_{\lambda_N}+1)]|\lambda_N\} - \lambda_N E\{[f_{x_0}(U_{\lambda_N}+1)]|\lambda_N\}\}| \\ &= |E\{(\lambda-\lambda_N)E[f_{x_0}(U_{\lambda_N}+1)|\lambda_N]\}| \\ &\leq \sup_{x\geq 1} f_{x_0}(x) E|\lambda_N - \lambda| \\ &\leq \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))E|\lambda_N-\lambda|}{(x_0+1)p_\lambda(x_0+1)}. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), the proof is complete. □

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 3.1

Corollary 3.1. *If $p_1 = p_2 = \cdots = p$, then we have the following:*

$$|\mathbb{PS}_N(x_0) - \mathbb{P}_\lambda(x_0)| \leq \min\left\{E(1-e^{-\lambda_N})qp^{-1}, \frac{E(N)q^2p^{-2}}{x_0}\right\} + \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))E|N-E(N)|qp^{-1}}{(x_0+1)p_\lambda(x_0+1)}. \quad (3.4)$$

4. Examples

We give two examples to illustrate the result in the case of X_i's are identically distributed.

Example 4.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N=k) = \begin{cases} \frac{1}{2}, & k = n, \\ \frac{1}{2}, & k = 2n, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore $E(N) = \frac{3n}{2}$ and $E|N - E(N)| = \frac{n}{2}$. Let $p_1 = p_2 = \cdots = p$, then for $x_0 \in \mathbb{N}$, we have

$$|\mathbb{PS}_N(x_0) - \mathbb{P}_\lambda(x_0)| \leq \min\left\{qp^{-1}, \frac{3nq^2p^{-2}}{2x_0}\right\} + \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))nqp^{-1}}{2(x_0+1)p_\lambda(x_0+1)}.$$

Example 4.2. Let N be a positive integer-valued random variable with probability function

$$P(N=n) = \frac{1}{2^n},\ n = 1, 2, ...,$$

then we have $E(N) = 2$ and $E|N - E(N)| = 1$. If $p_1 = p_2 = \cdots = p$, then for $x_0 \in \mathbb{N}$, we obtain

$$|\mathbb{PS}_N(x_0) - \mathbb{P}_\lambda(x_0)| \leq \min\left\{qp^{-1}, \frac{2q^2p^{-2}}{x_0}\right\} + \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))qp^{-1}}{(x_0+1)p_\lambda(x_0+1)}.$$

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