

THE LAURENT SERIES FOR THE QUATERNIONIC CASE

M.F. Borges¹ §, J. Maranhão²

¹UNESP - São Paulo State University
S.J. Rio Preto Campus
15054-000, São José do Rio Preto, BRAZIL

²Department of Mathematics
UFMA - Federal University of Maranhão
65085-580, Maranhão, BRAZIL

Abstract: The present work has the scope to show the Laurent Series for quaternionic functions. It will be shown that the Laurent Series for the Quaternionic Case is analogous to the textbook case of Complex Analysis [1]-[2] already well established.

AMS Subject Classification: 30G99, 30E99

Key Words: Laurent series, hypercomplex functions, quaternionic functions

1. Introduction

The theory of hypercomplex functions mainly for quaternions and octonions, plays an important role in Mathematical Analysis, Geometry, as well as in Theoretical Physics as it reveals underlying aspects of symmetry in unifying Theories [1]-[2]. In recent days this theory presented analogous formulation to some well known results in the Complex Analysis [2], e.g trigonometric functions in hypercomplex version, Taylor Series and a Cauchy-Like Theorem for Quaternionic and Octonionic Functions [3], [8].

This work will be divided as follows:

Received: May 7, 2013

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§Correspondence author

- The Cauchy Like Theorem for Quaternionic Functions;
- Laurent's Theorems for Complexa Variables;
- Laurent's Theorems for Quaternionic Functions.

The Cauchy-Like Theorem for Quaternionic Functions will be the key for determining the coefficients of the Laurent Series of the Qaternionic type and it is present below:

Theorem 1. *Let Ω be a domain single sign in four-dimensional space and $f(q)$ be a regular function on Ω then*

$$\int_{\varphi} \frac{f(q)}{q - q_0} dq = \pi(i + j + 2k)f(q_0), \quad (1)$$

where φ is a simple closed hypersurface in Ω and $q_0 \in \Omega$.

The Laurent series in the classical case is determined according to the following theorem [1] e [2]:

Theorem 2. *If the annulus (C_1, C_2) of center z_0 and understood in a void area (C_1, C_2) in which the function $f(z)$ is analytic in the crown that is representable by a power series positive and negative $z - z_0$, call Laurent Series*

$$f(z) = \sum_{n=-} a_n(z - z_0)^n,$$

whose coefficients, called Laurent coefficients are given by the Cauchy's formula:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

2. The Laurent Series for Quaternionic Functions

In order to fix ideas it is important to mention on the region that we will work the idea of Laurent Series for Quaternionic Case. The complex case predicts that the region will be crafted as a flat region. Thus, in the quaternionic case the region is conjectured as region of the four-dimensional space. Let us now assert the following theorem:

Theorem 3. *Be the region (C_1, C_2) in four-dimensional space centered in q_0 and understood in a domain (C_1, C_2) in which function $f(q)$ is an analytical, so this is representable in the region concerned by a power series of positive and negative $q - q_0$, call Laurent Series*

$$f(q) = \sum_{n=-} a_n(q - q_0)^n$$

and whose coefficients, called Laurent coefficients are given by the Cauchy formula for quaternionic functions [8]:

$$a_n = \frac{1}{\pi(i + j + 2k)} \oint_C \frac{f(q) dq}{(q - q_0)^{n+1}}$$

Demonstration: The proof is analogous to the complex case. Thus using Taylor's theorem we have that:

$$f(q) = \frac{1}{\pi(i + j + 2k)} \oint_{C'_1} \frac{f(t) dt}{t - q} - \frac{1}{\pi(i + j + 2k)} \oint_{C'_2} \frac{f(t) dt}{t - q}$$

assuming that the surface C_2 is traversed in the positive direction, C_1 in the negative sense. Thus, as in the Taylor series, we have to C_1

$$\frac{1}{\pi(i + j + 2k)} \oint_{C'_1} \frac{f(t) dt}{t - q} = \sum_{n=0} a_n(q - q_0)^n$$

whose coefficients are given by:

$$a_n = \frac{f^n(q_0)}{n!} = \frac{1}{\pi(i + j + 2k)} \oint_{C'_1} \frac{f(t) dt}{(t - q_0)^{n+1}}.$$

Now follows a case in which the integration is performed by C_2 . Therefore, we write:

$$\frac{1}{t - q} = -\frac{1}{q - q_0} \left(1 - \frac{t - q_0}{q - q_0}\right)^{-1},$$

and considering that now:

$$\left| \frac{t - q_0}{q - q_0} \right| < 1,$$

follows immediately that

$$\left(1 - \frac{t - q_0}{q - q_0}\right)^{-1} = \sum_{m=0} \left(\frac{t - q_0}{q - q_0}\right)^m.$$

Similarly to the previous case, we can do:

$$-\frac{1}{\pi(i+j+2k)} \oint_{C'_2} \frac{f(t)}{t-q} = \sum_{n=-1}^{\infty} (q-q_0)^n \frac{1}{\pi(i+j+2k)} \oint_{C'_2} \frac{f(t)}{(t-q_0)^{n+1}}$$

which gives us:

$$f(z) = \sum_{n=0}^{\infty} a_n (q-q_0)^n + \sum_{n=-1}^{\infty} a_n (q-q_0)^n$$

with $n > 0$ is a valid expression:

$$a_n = \frac{1}{\pi(i+j+2k)} \oint \frac{f(t)dt}{(t-q_0)^{n+1}}$$

3. Conclusion

In this brief report the Laurent Series for Quaternionic Functions have been determined it will be important in quaternionic functions that will be addressed in future works, for instance:

- Deepening Integration Theory of Functions Hypercomplex, quaternionic functions in particular;
- Formulation of the Theory of Functions Waste for Quaternions;
- Implementation and resolution of Quadruples Improper integrals.

The topics listed above will also be key for solving physical problems which comes from a formulation in four dimensions, and support in problems relating to the consolidation of the Theory of Hypercomplex Functions, a goal of the research so far.

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