International Journal of Pure and Applied Mathematics
Volume 91 No. 1 2014, 131-134
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i1.13

ijpam.eu

More on the Diophantine equation $3^x + 85^y = z^2$

Banyat Sroysang
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Rangsit Center
Pathumthani, 12121, THAILAND

Abstract: In this paper, we show that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 85^y = z^2$ where x, y and z are non-negative integers. This result implies that $(1, 0, 0, 2)$ is a solution (x, u, v, z) for the Diophantine equation $3^x + 5^u 17^v = z^2$.

AMS Subject Classification: 11D61
Key Words: exponential Diophantine equation

1. Introduction

In [9], Sroysang showed that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions (x, y, z) for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers.

In [8], Rabago solved the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where x, y and z are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.

In [11, 12], Sroysang showed that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for both two Diophantine equations $3^x+5^y = z^2$ and $3^x+17^y = z^2$ where x, y and z are non-negative integers. We note that $5 \times 17 = 85$. In

Received: January 14, 2014

this paper, we will solve the Diophantine equation $3^x + 85^y = z^2$ where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$. Moreover, we obtain that $(1, 0, 0, 2)$ is a solution (x, u, v, z) for the Diophantine equation $3^x + 5^u 17^v = z^2$.

For other results, we refer to [1, 2, 3, 4, 6, 7, 10, 13, 14, 15, 16, 17].

2. Preliminaries

Proposition 2.1. [5] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [11] $(1, 2)$ *is a unique solution* (x, z) *for the Diophantine equation* $3^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1 + 85^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 85^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. This implies that $y \geq 1$. Thus, $z^2 = 1 + 85^y \geq 1 + 85^1 = 86$. We obtain that $z \geq 10$. Now, we consider on the equation $z^2 - 85^y = 1$. By Proposition 2.1, we have $y = 1$. This implies that $z^2 = 86$. This is a contradiction. Hence, the Diophantine equation $1 + 85^y = z^2$ has no non-negative integer solution. □

3. Main Results

Theorem 3.1. $(1, 0, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $3^x + 85^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $3^x + 85^y = z^2$. Since z is even, we obtain that $z^2 \equiv 0 \pmod 4$. Note that $85^y \equiv 1 \pmod 4$. This implies that $3^x \equiv 3 \pmod 4$. By Lemma 2.3, we have $x \geq 1$. It follows that x is odd. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 2$.

Case $y \geq 1$. Note that $85^y \equiv 0 \pmod 5$. Since $3^y \equiv 2 \pmod 5$ or $3^y \equiv 3 \pmod 5$, we obtain that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This implies that z is even. This is a contradiction.

Hence, $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 85^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $3^x + 85^y = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $3^x + 85^y = w^4$. Let $z = w^2$. This implies that $3^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. This implies that $w^2 = z = 2$. This is a contradiction. Hence, the Diophantine equation $3^x + 85^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers. □

Corollary 3.3. $(1, 0, 0, 2)$ *is a solution* (x, u, v, z) *for the Diophantine equation* $3^x + 5^u 17^v = z^2$.

Proof. This follows from Theorem 3.1 where $y = u = v$. □

References

[1] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, **15** (2007), 145−148.

[2] S. Chotchaisthit, On the Diophantine equation $2^x + 11^y = z^2$, *Maejo Int. J. Sci. Technol.*, **7** (2013), 291−293.

[3] S. Chotchaisthit, On the Diophantine equation $4^x + p^y = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, **1** (2012), 191−193.

[4] S. Chotchaisthit, On the Diophantine equation $p^x + (p+1)^y = z^2$ where p is a Mersenne prime, *Int. J. Pure Appl. Math.*, **88** (2013), 169−172.

[5] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167−195.

[6] J. F. T. Rabago, More on the Diophantine equation of type $p^x + q^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 15−16.

[7] J. F. T. Rabago, On an open problem by B. Sroysang, *Konuralp J. Math.*, **1** (2013), 30−32.

[8] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 28−29.

[9] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133–137.

[10] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601–604.

[11] B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605–608.

[12] B. Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 111–114.

[13] B. Sroysang, On the Diophantine equation $5^x + 23^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 119–122.

[14] B. Sroysang, On the Diophantine equation $7^x + 8^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111–114.

[15] B. Sroysang, On the Diophantine equation $47^x + 49^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 279–282.

[16] B. Sroysang, On the Diophantine equation $89^x + 91^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 283–286.

[17] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.*, **1** (2011), 25–28.