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ON THE GENERALIZED HYERS-ULAM STABILITY FOR EULER-LAGRANGE TYPE FUNCTIONAL EQUATION

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Abstract: In this paper we give the general solution of the quadratic functional equation

$$f(x+3y)+f(y+3z)+f(z+3x)-3f(x+y+z)=7\big(f(x)+f(y)+f(z)\big),$$

and investigate its generalized Hyers-Ulam-Rassias stability.

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1. Introduction

In 1940, S.M. Ulam [14] raised the question concerning the stability of group homomorphisms:

Let G be a group and let G' be a metric group with metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \longrightarrow G'$ satisfies

$$d(f(xy), f(x)f(y)) < \delta \quad \text{for all} \quad x, y \in G,$$

then there exists a homomorphism $F : G \longrightarrow G'$ with

$$d(f(x), F(x)) < \varepsilon \quad \text{for all} \quad x \in G\ ?$$

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In [2], Hyers considered the case of approximately additive mappings $f : X \longrightarrow Y$, where X and Y are Banach spaces and f satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$. It was shown that the limit

$$F(x) = \lim_{n \longrightarrow \infty} 2^{-n} f(2^n x),$$

exists for all $x \in X$ and that $F : X \longrightarrow Y$ is the unique additive mapping satisfying

$$\|f(x) - F(x)\| \leq \varepsilon.$$

A generalization of Hyers theorem provided by Rassias in [4]. In 1982-1994, J. M. Rassias (see [5-12]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [13]. In 1994, a generalization of the Rassias' theorem was obtained by Gavruta [1] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Consider the following functional equations:

$$\begin{aligned} &f(x+2y) + f(y+2z) + f(z+2x) \\ &\qquad = 2f(x+y+z) + 3\big(f(x) + f(y) + f(z)\big), \quad (I) \end{aligned}$$

and

$$f(x+y) + f(y+z) + f(z+x) = f(x+y+z) + f(x) + f(y) + f(z). \quad (II).$$

Recently, the author investigated in his paper Zivari the generalized Hyers-Ulam stability of the equation (I), and the functional equation (II) was solved by Pl. Kannappan in [3].

In this paper we consider the quadratic functional equation

$$f(x+3y) + f(y+3z) + f(z+3x) = 3f(x+y+z) + 7\big(f(x) + f(y) + f(z)\big),$$

and determine the general solution of this functional equation and investigate its generalized Hyers-Ulam-Rassias stability.

2. The General Solution

We commence with the next result which is provide the general solution of the proposed functional equation.

Theorem 1. *Let X and Y be real vector spaces. A function $f : X \longrightarrow Y$ satisfies the functional equation*

$$\begin{aligned} &f(x+3y)+f(y+3z)+f(z+3x) \\ &\qquad = 3f(x+y+z)+7\big(f(x)+f(y)+f(z)\big), \end{aligned} \tag{1}$$

for all $x, y, z \in X$ if and only if it satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad (x, y \in X). \tag{2}$$

Proof. Assume that a function $f : X \longrightarrow Y$ satisfies (1). Take $x = y = z$ in (1), we get

$$f(4x)-f(3x)=7f(x), \tag{3}$$

for all $x \in X$, which implies that $f(0) = 0$. Letting $y = z = 0$ in (1), we have

$$f(3x)=9f(x), \tag{†}$$

Combining Eq.(†) and (3), we get

$$f(4x)=16f(x), \tag{‡}$$

Letting $z = 0$ in (1), we obtain

$$f(x+3y)+f(y)+f(3x)-3f(x+y)=7(f(x)+f(y)).$$

Applying Eq.(†), the above equation simplifies to

$$f(x+3y)-3f(x+y)=6f(y)-2f(x). \tag{4}$$

Replacing x by y and y by x in (4), so

$$f(y+3x)-3f(y+x)=6f(x)-2f(y). \tag{5}$$

Letting $y = z$ in (1), we get

$$f(x+3y)+f(4y)+f(y+3x)-3f(x+2y)=7\big(f(x)+2f(y)\big).$$

Using Eq.(‡), the above equation simplifies to

$$f(x+3y)+f(y+3x)-3f(x+2y)=7f(x)-2f(y). \tag{6}$$

Eliminating $f(x+3y)$ and $f(y+3x)$ from (6) by applying (4) and (5), we get

$$2f(x+y)+2f(y)=f(x)+f(x+2y).$$

Now the classical quadratic functional equation (2) follows if we replacing x by $x-y$ in above equation.

Conversely, suppose that a function $f: X \longrightarrow Y$ satisfies (2). Replacing x with $x+2y$ and all cyclic permutations of the variables in (2), then we have

$$\begin{aligned} f(x+3y)+f(x+y)&=2f(x+2y)+2f(y),\\ f(y+3z)+f(y+z)&=2f(y+2z)+2f(z),\\ f(z+3x)+f(z+x)&=2f(z+2x)+2f(x). \end{aligned} \tag{7}$$

Eliminating $f(x+2y)$, $f(y+2z)$ and $f(z+2x)$ in the sum of all equations in (7), by applying Theorem 2.1 of [15], then we get

$$\begin{aligned} &f(x+3y)+f(y+3z)+f(z+3x)-4f(x+y+z)=\\ &\qquad -\big(f(x+y)+f(y+z)+f(z+x)\big)+8\big(f(x)+f(y)+f(z)\big). \end{aligned} \tag{8}$$

Combining Eq. (II) and (8), then the functional equation (1) follows, and the proof is complete. □

For convenience, we use the following abbreviations:

$$\begin{aligned} Df(x,y,z)&=f(x+3y)+f(y+3z)+f(z+3x)\\ &\quad -3f(x+y+z)-7\big(f(x)+f(y)+f(z)\big). \end{aligned}$$

Theorem 2. *Suppose X is a real vector space and Y is a Banach space. Let $\varphi: X^3 \longrightarrow [0,\infty)$ be a function such that*

$$\lim_{n\longrightarrow\infty} 9^{-n}\varphi(3^n x, 3^n y, 3^n z)=0 \quad (x,y,z\in X), \tag{9}$$

and $\sum_{n=0}^{\infty} 9^{-n}\varphi(3^n x, 3^n y, 3^n z)$ be convergent. Let $f: X \longrightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$\|Df(x,y,z)\| \le \varphi(x,y,z), \tag{10}$$

for all $x,y,z\in X$, then there exists a unique function $F: X \longrightarrow Y$ which satisfies (1) and

$$\|f(x)-F(x)\| \le \frac{1}{9}\sum_{n=0}^{\infty} 9^{-n}\varphi(3^n x,0,0) \quad (x\in X). \tag{11}$$

Proof. Letting $y = z = 0$ in (10), we get

$$\|f(3x) - 9f(x)\| \leq \varphi(x, 0, 0).$$

Dividing the above inequality by 9, we obtain

$$\|\frac{f(3x)}{9} - f(x)\| \leq \frac{1}{9}\varphi(x, 0, 0). \tag{12}$$

One can use the induction on n to show that

$$\|\frac{f(3^n x)}{9^n} - f(x)\| \leq \frac{1}{9}\sum_{k=0}^{n-1} 9^{-k}\varphi(3^k x, 0, 0), \tag{13}$$

for all and all $x \in X$. Replacing x by $3^m x$ in (13), we have

$$\|\frac{f(3^{n+m}x)}{9^{n+m}} - \frac{f(3^m x)}{9^m}\| \leq \frac{1}{9}\sum_{k=m}^{n+m-1} 9^{-k}\varphi(3^k x, 0, 0) \quad (x \in X).$$

It follows that the sequence $\{\frac{1}{9^n} f(3^n x)\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{9^n} f(3^n x)\}$ is convergent. Set

$$F(x) := \lim_{n \longrightarrow \infty} \frac{1}{9^n} f(3^n x), \qquad (x \in X).$$

Then by the definition of F, we can see that (11) holds. To show that F satisfies in (1), we set $(x, y, z) = (3^n x, 3^n y, 3^n z)$ in (10), and divide the result by 9^n, hence

$$\frac{1}{9^n}\|Df(3^n x, 3^n y, 3^n z)\| \leq \frac{\varphi(3^n x, 3^n y, 3^n z)}{9^n}.$$

Take the limit as $n \longrightarrow \infty$, so

$$\|DF(x, y, z)\| \leq 0,$$

for all $x, y, z \in X$. Therefore, F satisfies (1). The uniqueness of F follows from Theorem 1. □

Corollary 3. *Let $f : X \longrightarrow Y$ be a function with $f(0) = 0$ and*

$$\|Df(x, y, z)\| \leq \varepsilon,$$

for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique function $F : X \longrightarrow Y$ which satisfies (1) and

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{8} \quad (x \in X).$$

Proof. Apply Theorem 2, for $\varphi(x,y,z)=\varepsilon$. $\square$

Corollary 4. *Let $f : X \longrightarrow Y$ be a function with $f(0)=0$ and*

$$\|Df(x,y,z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

with $p<2$ and for some $\varepsilon>0$ and for all $x,y,z\in X$. Then there exists a unique quadratic function $F : X \longrightarrow Y$ such that

$$\|f(x)-F(x)\| \leq \frac{\varepsilon}{|9-3^p|}\|x\|^p \qquad (x\in X).$$

Proof. Apply Theorem 2, for $\varphi(x,y,z)=\varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$. $\square$

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