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SOME PROPERTIES OF THE EULER SUMMABILITY METHOD IN COMPLETE ULTRAMETRIC FIELDS

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Abstract: In this paper, we study some properties of the Euler method of summability in complete, non-trivially valued, ultrametric fields of characteristic zero and prove few Tauberian theorems of Euler method on such a field.

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Key Words: complete ultrametric fields, regular summability methods, Euler method, translative methods, Tauberian theorem

1. Introduction and Preliminaries

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric field of characteristic zero (Q_p, the p-adic field for a prime p, is one such field). Infinite matrices, sequences and series considered in the sequel have entries in K. Given an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \cdots$ and a sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \ldots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $Ax = (Ax)_n$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$, $n = 0, 1, 2, \cdots$, it being assumed that the series on the right converge. If $\{(Ax)_n\}$

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converges to s, we say that $x = \{x_k\}$ is summable A or A-summable to s. If $\lim_{n\to\infty}(Ax)_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The following theorem, which gives necessary and sufficient conditions for $A = (a_{nk})$ to be regular in terms of the entries of the matrix, is well-known (see [4] for a proof using 'Uniform Boundedness Principle' and [5] for a proof using 'Sliding Hump Method').

Theorem 1.1. *$A = (a_{nk})$ is regular if and only if*

(i) $\sup_{n,k} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \ldots$;*and*

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$.

An infinite series $\sum_{k=0}^{\infty} x_k$, $x_k \in k$, $k = 0, 1, 2, \cdots$ is said to be A-summable to s if $\{s_n\}$ is A-summable to s, where $s_n = \sum_{k=0}^{n} x_k$, $n = 0, 1, 2, \cdots$.

In the present paper, we prove some interesting properties of the Euler method of summability introduced earlier by Natarajan [9].

General references for the study of summability methods in the classical case are [3], [10], while, for analysis in ultrametric fields, see [2].

Definition 1.1. Let $r \in K$ be such that $|1 - r| < 1$. The Euler method of order r or the (E, r) method is given by the infinite matrix $e_{nk}(r)$ which is defined as follows:

If $r \neq 1$, then $e_{nk}{}^{(r)} = \begin{cases} {}^{n}c_k r^k (1-r)^{n-k}, & \text{for} \quad k \leq n \\ 0, \quad \text{for} \quad k > n \end{cases}$

$(e_{nk}^{(r)})$ is called the (E, r) matrix.

Remark 1.1. Note that $r \neq 0$, since $|1 - r| < 1$.

The following results are needed in the sequel.

Theorem 1.2. *([9], Theorem 1.2) The (E, r) method is regular.*

Theorem 1.3. *([9], Theorem 1.3) $(e_{nk}^{(r)})(e_{nk}^{(s)})$ is the (E, rs) matrix.*

Corollary 1.1. *([9], Corollary 1.4) The (E, r) matrix is invertible and its inverse is the $\left(E, \frac{1}{r}\right)$ matrix.*

2. Main Results

In this section, we prove some interesting properties of the Euler method.

Theorem 2.1 (Limitation theorem). *If $\sum_{k=0}^{\infty} x_k$ is (E,r) summable, then $\{x_k\}$ is bounded.*

Proof. Let $\{\sigma_n^{(r)}\}$ be the (E,r) transform of $\{s_n\}$, where $s_n = \sum_{k=0}^{n} x_k$, $n = 0,1,2,\dots$, i.e.,

$$\sigma_n^{(r)} = \sum_{k=0}^{n} {}^n c_k r^k (1-r)^{n-k} s_k, \quad n = 0,1,2,\dots.$$

By hypothesis $\lim_{n\to\infty} \sigma_n = \sigma$ (say). So $\{\sigma_n\}$ is bounded, i.e., there exists $M > 0$ such that $|\sigma_n| \le M$, $n = 0,1,2,\dots$. Note that, in view of Corollary (1.1),

$$s_n = \sum_{k=0}^{n} {}^n c_k \left(\frac{1}{r}\right)^k \left(1 - \frac{1}{r}\right)^{n-k} \sigma_k, \quad n = 0,1,2,\dots.$$

$$\text{Thus } |s_n| \le \max_{0\le k\le n} |{}^n c_k| \frac{1}{|r|^k} \frac{|r-1|^{n-k}}{|r|^{n-k}} |\sigma_k| \le M,$$

since $|{}^n c_k| \le 1$, $|r-1| < 1$, $|r| = |(r-1)+1| = \max(|r-1|, 1) = 1$. Consequently, $|x_k| = |s_k - s_{k-1}| \le \max(|s_k| - |s_{k-1}|) \le M$, $k = 0,1,2,\dots$ so that $\{x_k\}$ is bounded. □

Remark 2.1. We recall that the classical Mazur-Orlicz theorem says that if a conservative matrix sums a bounded divergent sequence, then it sums an unbounded one. It was pointed out in [8] that the above theorem fails to hold in the ultrametric case, a counterexample being any regular (N, p_n) method. Theorem (**??**) shows that any (E,r) method is also a counter example to show that the Mazur-Orlicz theorem fails to hold in the ultrametric set up.

Definition 2.1. Given a sequence $\{x_k\}$, define the sequence $\{\bar{x}_k\}$ by

$$\bar{x}_0 = 0, \quad \bar{x}_k = x_{k-1}, \quad k \ge 1.$$

$A = (a_{nk})$ is said to be left translative if the A-summability of $\{x_k\}$ to s implies the A-summability of $\{\bar{x}_k\}$ to s. A is said to be right translative if the A-summability of $\{\bar{x}_k\}$ to s implies the A-summability of $\{x_k\}$ to s. If A is both left and right translative, A is said to be translative.

Theorem 2.2. *(E,r) is translative.*

Proof. Let $\{\sigma_n(r)\}$ be the (E,r) transform of $\{x_k\}$ and $\{\tau_n(r)\}$ be the (E,r) transform of $\{\bar{x}_k\}$. We shall now prove that

$$\sigma_n(r) = \left(1 - \frac{1}{r}\right)\tau_n(r) + \frac{1}{r}\tau_{n+1}(r), \tag{2.1}$$

$$\begin{aligned} i.e., \sum_{k=0}^{n} {}^{n}c_k r^k (1-r)^{n-k} x_k &= \left(1 - \frac{1}{r}\right) \sum_{k=0}^{n} {}^{n}c_k r^k (1-r)^{n-k} \bar{x}_k \\ &\quad + \frac{1}{r} \sum_{k=0}^{n+1} {}^{(n+1)}c_k r^k (1-r)^{n+1-k} \bar{x}_k, \\ i.e., \sum_{k=0}^{n} {}^{n}c_k r^k (1-r)^{n-k} x_k &= \left(1 - \frac{1}{r}\right) \sum_{k=1}^{n} {}^{n}c_k r^k (1-r)^{n-k} x_{k-1} \\ &\quad + \frac{1}{r} \sum_{k=1}^{n} {}^{(n+1)}c_k r^k (1-r)^{n+1-k} x_{k-1}, \text{since } \bar{x}_0 = 0 \end{aligned}$$

For $0 \le j \le n$, coefficient of x_j on the left side of (2.1) $= {}^{n}c_j r^j (1-r)^{n-j}$. Also coefficient of x_j on the right side of (2.1)

$$\begin{aligned} &= \left(1 - \frac{1}{r}\right) {}^{n}c_{(j+1)} r^{j+1} (1-r)^{n-j-1} + \frac{1}{r} {}^{(n+1)}c_{(j+1)} r^{j+1} (1-r)^{n-j} \\ &= -\frac{1-r}{r} {}^{n}c_{(j+1)} r^{j+1} (1-r)^{n-j-1} + \frac{1}{r} {}^{(n+1)}c_{(j+1)} r^{j+1} (1-r)^{n-j} \\ &= -\frac{1}{r} {}^{n}c_{(j+1)} r^{j+1} (1-r)^{n-j} + \frac{1}{r} {}^{(n+1)}c_{(j+1)} r^{j+1} (1-r)^{n-j} \\ &= - {}^{n}c_{(j+1)} r^{j} (1-r)^{n-j} + {}^{(n+1)}c_{(j+1)} r^{j} (1-r)^{n-j} \\ &= \left\{ {}^{(n+1)}c_{(j+1)} - {}^{n}c_{(j+1)} \right\} r^j (1-r)^{n-j} \\ &= {}^{n}c_j r^j (1-r)^{n-j}. \end{aligned}$$

Thus (2.1) holds. Suppose $\lim_{n\to\infty} \tau_n(r) = s$. Taking limit as $n \to \infty$ in (2.1), we see that

$$\lim_{n\to\infty} \sigma_n(r) = \left(1 - \frac{1}{r}\right) s + \frac{1}{r} s = s,$$

so that (E,r) is right translative. Now,

$$\tau_n(r) = \sum_{k=0}^{n} {}^{n}c_k r^k (1-r)^{n-k} \bar{x}_k$$

$$= \sum_{k=1}^{n} {}^{n}c_k r^k (1-r)^{n-k} x_{k-1}, \quad \text{since } \bar{x}_0 = 0$$
$$= \sum_{j=0}^{n-1} {}^{n}c_{(j+1)} r^{j+1} (1-r)^{n-j-1} s_j$$
$$= \sum_{j=0}^{n-1} {}^{n}c_{(j+1)} r^{j+1} (1-r)^{n-j-1} \left(\sum_{k=0}^{j} {}^{j}c_k \left(\frac{1}{r}\right)^k \left(1 - \frac{1}{r}\right)^{j-k} \sigma_k(r) \right)$$

$$= \sum_{k=0}^{n-1} r(1-r)^{n-k-1} \sigma_k(r) \left(\sum_{j=k}^{n-1} (-1)^{j-k} \; {}^{n}c_{(j+1)} \; {}^{j}c_k \right).$$

That is $\tau_n(r) = \sum_{k=0}^{n-1} r(1-r)^{n-k-1} \sigma_k(r) \left(\sum_{j=k}^{n-1} (-1)^{j-k} \; {}^{n}c_{(j+1)} \; {}^{j}c_k \right)$.

Using the identity $\sum_{k=0}^{n-1} \left(\sum_{j=k}^{n-1} (-1)^{j-k} \; {}^{n}c_{(j+1)} \; {}^{j}c_k \right) z^k = \sum_{k=0}^{n-1} z^k$, we see that

$$\sum_{j=k}^{n-1} (-1)^{j-k} \; {}^{n}c_{(j+1)} \; {}^{j}c_k = 1, \quad 0 \leq k \leq (n-1). \tag{2.2}$$

In view of (2.2), we have, $\tau_n(r) = \sum_{k=0}^{n-1} r(1-r)^{n-k-1} \sigma_k(r)$.

Since $|1-r| < 1$, all the conditions of theorem (1.1) are fulfilled and so $\lim_{k\to\infty} \sigma_k(r) = s$ implies that $\lim_{n\to\infty} \tau_n(r) = s$. Thus (E, r) is left translative. This completes the proof of the theorem. □

Definition 2.2. The infinite matrix methods $A = (a_{nk})$, $B = (b_{nk})$ are said to be ‘consistent’ if no sequence is summable to different values by A and B, i.e., if a sequence $\{x_n\}$ is A summable to ℓ and B summable to m, then $\ell = m$.

As in the case of regular (N, p_n) methods (see [11], Theorem 4.1), we have the following result.

Theorem 2.3. *Any two Euler methods are consistent.*

Proof. Consider the Euler methods (E,r) and (E,s). We then have $|1-r|$, $|1-s|<1$. Let $\{\sigma_n(r)\}$, $\{\tau_n(s)\}$ be the (E,r), (E,s) transforms of $\{x_n\}$ respectively. Let $\lim_{n\to\infty}\sigma_n(r)=\sigma$ and $\lim_{n\to\infty}\tau_n(s)=\tau$. We claim that $\sigma=\tau$.

Now, $\sigma_n(r)=(E,r)(\{x_n\})$ and $\tau_n(s)=(E,s)(\{x_n\})$.

$$\begin{aligned} \text{So}\quad \sigma_n(r) &= (E,r)(E,s)^{-1}(\tau_n(s)) \\ &= ((E,r)\left(E,\frac{1}{s}\right)(\tau_n(s)), \text{using Corollary (1.1)} \\ &= \left(E,\frac{r}{s}\right)(\tau_n(s)), \text{using theorem (1.3)} \end{aligned} \tag{2.3}$$

$$\begin{aligned} \text{Note that}\left|1-\frac{r}{s}\right| = \left|\frac{s-r}{s}\right| &= |s-r|,\quad \text{since } |s|=1,\ \text{using } |1-s|<1 \\ &= |(1-r)-(1-s)| \\ &\le \max(|(1-r)|,|(1-s)|) \\ &< 1, \end{aligned}$$

so that $\left(E,\frac{r}{s}\right)$ is regular, in view of definition (1.1) and theorem (1.2). Using (2.3), it follows that $\sigma=\tau$, completing the proof. □

Remark 2.2. In view of Theorem 2.3, we are able to define a parameterless Euler method E of summability as follows.
A sequence $\{x_n\}$ is summable E to σ if there exists $r\in K$, $|1-r|<1$ such that $\{x_n\}$ is (E,r) summable to σ.

We shall now prove a few Tauberian theorems for the method (E,r) modelled on those proved for (N,p_n) methods by Natarajan [7].

Theorem 2.4. *If* $\sum_{k=0}^{\infty}a_k$ *is* (E,r) *summable to* σ *and if* $a_n\to\ell$, $n\to\infty$, *then* $\sum_{k=0}^{\infty}a_k$ *converges to* σ.

Proof. In view of theorem (1) of [7], it suffices to prove that the sequence $\{k\}$ of integers is not (E,r) summable. Let $\{\sigma_n(r)\}$ be the (E,r) transform of $\{k\}$, i.e., $\sigma_n(r)=\sum_{k=0}^{n}{}^{n}c_k r^k(1-r)^{n-k}k$, $n=0,1,2,\ldots$

$$\begin{aligned} \text{Now, } \sigma_{n+1}(r)-\sigma_n(r) &= \sum_{k=0}^{n+1}{}^{(n+1)}c_k r^k(1-r)^{n+1-k}k-\sum_{k=0}^{n}{}^{n}c_k r^k(1-r)^{n-k}k \\ &= \sum_{k=1}^{n+1}{}^{(n+1)}c_k r^k(1-r)^{n+1-k}k-\sum_{k=1}^{n}{}^{n}c_k r^k(1-r)^{n-k}k \end{aligned}$$

$$= \sum_{k=1}^{n} {}^{(n+1)}c_k r^k (1-r)^{n+1-k} k + r^{n+1}(n+1)$$
$$- \sum_{k=1}^{n-1} {}^{n}c_k r^k (1-r)^{n-k} k - r^n n$$

Using $|1-r| < 1$, $|r| = 1$, $|k| \leq 1$, $k = 0, 1, 2, \ldots$, we have,

$$\left| \sum_{k=1}^{n} {}^{(n+1)}c_k r^k (1-r)^{n+1-k} k \right| \leq \underset{1 \leq k \leq n}{Max} \left| {}^{(n+1)}c_k \right| |r|^k |1-r|^{n+1-k} |k|$$
$$< 1;$$

Similarly, $\left| \sum_{k=1}^{n-1} {}^{n}c_k r^k (1-r)^{n-k} k \right| < 1$;

$$\begin{aligned} \left| r^{n+1}(n+1) - r^n n \right| &= \left| n r^n (r-1) + r^{n+1} \right| \\ &= Max \left\{ |n||r|^n |r-1|, |r|^{n+1} \right\} \\ &= 1, \end{aligned}$$

so that $|\sigma_{n+1}(r) - \sigma_n(r)| = 1, \quad n = 0, 1, 2, \cdots$ Thus $\{\sigma_n(r)\}$ is not a Cauchy sequence and hence diverges, i.e., $\{k\}$ is not (E, r) summable, completing the proof. □

Using Theorem (3) of [7], we have,

Theorem 2.5. *If $\sum_{k=0}^{\infty} a_k$ is (E, r) summable to σ and if $a_{n+1} - a_n \to \ell$, $n \to \infty$, then $\sum_{k=0}^{\infty} a_k$ converges to σ.*

As in the case of regular (N, p_n) methor ([7], Theorem 5), we have the following theorem too.

Theorem 2.6. *If $\sum_{k=0}^{\infty} a_k$ is (E, r) summable, then the following Tauberian conditions are equivalent:*

(i) $a_n \to \ell$, $n \to \infty$;

(ii) $a_{n+1} - a_n \to \ell'$, $n \to \infty$.

If, further, $a_n \neq 0$, $n = 0, 1, 2, \ldots$, each of

(iii) $\dfrac{a_{n+1}}{a_n} \to 1$, $n \to \infty$; *and*

(iv) $\dfrac{a_{n+2} + a_n}{a_{n+1}} \to 2$, $n \to \infty$

is a weaker Tauberian condition for the (E, r) summability of $\sum_{k=0}^{\infty} a_k$.

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