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ON THE DIOPHANTINE EQUATION $46^x + 64^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $46^{\mathsf{x}} + 64^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In 1844, Catalan [2] conjectures that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^{\mathsf{x}} - b^{\mathsf{y}} = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. In 2004, the conjecture was proved by Mihailescu [6]. For related equations, we refer to [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18].

In 2013, Sroysang [14] showed that the Diophantine equation $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Note that $23 = 2(10) + 3$ and $32 = 3(10) + 2$. Similarly, $46 = 4(10) + 6$ and $64 = 6(10) + 4$.

In this paper, we will show that the Diophantine equation $46^{\mathsf{x}} + 64^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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2. Preliminaries

Proposition 2.1. [6] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $46^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x and z such that $46^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 46^x + 1 \geq 46^1 + 1 = 47$. Then $z \geq 7$. Now, we consider on the equation $z^2 - 46^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 47$. This is a contradiction. Hence, the equation $46^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers. □

Lemma 2.3. *The Diophantine equation* $1 + 64^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 64^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 64^y \geq 1 + 64^1 = 65$. Then $z \geq 9$. Now, we consider on the equation $z^2 - 64^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 65$. This is a contradiction. Hence, the equation $1 + 64^y = z^2$ has no non-negative integer solution where y and z are non-negative integers. □

3. Results

Theorem 3.1. *The Diophantine equation* $46^x + 64^y = z^2$ *has no non-negative integer solution where* x, y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $46^x + 64^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. This implies that z is even. Then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. We note that $46^x \equiv 1 \pmod 3$ and $64^x \equiv 1 \pmod 3$. This implies that $z^2 \equiv 2 \pmod 3$. This is a contradiction. Hence, the equation $46^x + 64^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $46^x + 64^y = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. We set $z = w^2$. By Theorem 3.1, the equation $46^{\mathsf{x}} + 64^{\mathsf{y}} = z^2$ has no non-negative integer solution. Hence, the equation $46^{\mathsf{x}} + 64^{\mathsf{y}} = w^4$ has no non-negative integer solution where x, y and w are non-negative integers. □

4. Open Problem

Let $a, b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If $m = a(10) + b$ and $n = b(10) + a$, then we may ask what is the set of all solutions (x, y, z) for the Diophantine equation $m^{\mathsf{x}} + n^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers

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