International Journal of Pure and Applied Mathematics
Volume 92 No. 4 2014, 565-567
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v92i4.10
ijpam.eu

POSTULATION OF UNIONS OF TANGENT VECTORS

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: I collect here a few remarks on the postulation of sufficiently general unions of tangent vectors in linear systems on projective spaces and smooth quadric surface (in positive characteristic).

AMS Subject Classification: 14N05, 14H99
Key Words: postulation, Hilbert function, tangent vector, union of tangent vectors

1. Introduction

Let X be a integral smooth scheme. A tangent vector v of X is a connected degree two zero-dimensional scheme. In characteristic zero general unions of tangent vectors of X impose the maximal possible number of independent conditions to any linear system on X ([3], [2, Lemma 1.4]). In positive characteristic this is false, in general, but it is often true. In particular it is true in many interesting cases. We collect in this note all the cases we will need in [1].

We work over an algebraically closed field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$.

For any projective scheme X, any line bundle L on X, any linear subspace $V \subseteq H^0(X, L)$ and any closed subscheme $Z \subset X$ set $V(-Z) := V \cap H^0(X, L)$.

Received: January 14, 2014

Proposition 1. *Fix positive integers r, k, z. Let $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^r}(k))$ be any linear subspace. Assume $p > k$. Let $Z \subset \mathbb{P}^r$ be a general union of z tangent vectors. Then* $\dim(V(-Z)) = \max\{0, \dim(V) - 2z\}$.

Proposition 2. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix integers $z > 0$, $b \geq 0$, and $a \geq 0$. Assume $p > \max\{a, b\}$. Let $V \subseteq H^0(Q, \mathcal{O}_Q(a, b))$ be a linear subspace. Let $Z \subset Q$ be a general union of z tangent vectors of Q. Then* $\dim(V(-Z)) = \max\{0, \dim(V) - 2z\}$.

Lemma 1. *Fix an integer $k > 0$ and a linear subspace $V \subset \mathbb{P}^r$, $r \geq 1$, such that the base locus of V has codimension at least two. If p does not divide k, then* $\dim(V(-Z)) = \max\{\dim(V) - 2, 0\}$ *for a general tangent vector $Z \subset \mathbb{P}^r$.*

Proof. Let E be the base locus of V. Since $\dim(E) \leq 2$, there is a line $L \subseteq \mathbb{P}^r$ such that $V \cap L = \emptyset$. Hence $V|L$ induces a degree k morphism ϕ from L into a projective space. If p does not divide p, then ϕ is separable and hence $\dim(V_{|L}(-Z)) = \max\{0, \dim(V_{|L}) - 2\}$ for a general tangent vector Z of L. □

Proof of Proposition 2. By induction on z we reduce to the case $z = 1$. Decreasing if necessary either a or b or both we reduce to the case in which the base locus B of V contains no divisor of Q. If $b = 0$, then the result is obvious. Hence we may assume $b > 0$. Fix a general $L \in |\mathcal{O}_Q(1, 0)|$. Since L is general, we have $B_{\text{red}} \cap L = 0$. Hence $V_{|L}$ induces a degree b morphism ϕ from L into a projective space. Since $p > b$, ϕ is separable. Hence it is sufficient to take a general tangent vector of L. □

Proof of Proposition 1. We immediately reduce to the case $z = 1$. Let $\mathcal{B}$ be the base locus of V and $a \geq 0$ the degree of the codimension one part of $\mathcal{B}$. Let $\phi : \mathbb{P}^r \setminus \mathcal{B}_{\text{red}} \to \mathbb{P}^x$, $x := \dim(V) - 1$, be the morphism induced by V. The proposition is false for V and the integer $z = 1$ if and only if ϕ is not separable. Assume that ϕ is not separable. We get that p divides $k - a$ (Lemma 1), contradicting the assumption $p > k$. □

Proposition 3. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix integers $z > 0$ and $b \geq a \geq 0$. Let $Z \subset Q$ be a general union of z tangent vectors of Q. Then either $h^0(\mathcal{I}_Z(a, b)) = 0$ or $h^1(\mathcal{I}_Z(a, b)) = 0$.*

Proof. We may assume $a \geq b$. The case $a = 0$ is obvious. Hence we may assume $a > 0$ and use induction on the integer a. It is sufficient to do the case $z = \lfloor(a + 1)(b + 1)/2$ and $z = \lceil(a + 1)(b + 1)/2\rceil$. In particular we may assume $z \geq \lceil(a + 1)/2\rceil$. First assume a odd. Let $L \subset Q$ be a line of type $(1, 0)$. Take any $S \subset L$ such that $\sharp(S) = (a + 1)/2$. Let $W \subset L$ be the union of the tangent vectors of L with $W_{\mathrm{red}} = S$. We have $h^i(L, \mathcal{I}_W(b,a)) = 0$, $i = 0,1$. Hence to prove the proposition for the pair (a,b) it is sufficient to find a solution $Z' \subset Q \setminus L$ for the pair $(a, b-1)$ and then take $Z' \cup W$. Now assume a even. Take $A \subset L$ such that $\sharp(A) = a/2$ and a general $P \in L \setminus A$. Let $B \subset L$ be the union of the tangent vectors of L with $B_{\mathrm{red}} = A$ and v a general tangent vector of Q with $v_{\mathrm{red}} = \{P\}$. We have $(B \cup v) \cap L = B \cup \{P\}$ and $\mathrm{Res}_L(B \cup v) = \{P\}$. Since $h^i(L, \mathcal{I}(B \cup v) \cap L(a,b)$ and $\mathrm{Res}_L(B \cup v) = \{P\}$, to prove the proposition it is sufficient to prove that either $h^0(\mathcal{I}_{\{P\}\cup Z_1}(a, b-1)) = 0$ or $h^0(\mathcal{I}_{\{P\}\cup Z_1}(a, b-1)) = 0$ for a general union $Z_1 \subset Q$ of $z - 1 - a/2$ tangent vectors. For general L and $P \in L$, P may be considered as a general point of Q. Hence it is sufficient to prove that either $h^0(\mathcal{I}_{Z_1}(a, b-1)) = 0$ or $h^0(\mathcal{I}_{Z_1}(a, b-1)) = 0$. This is true by the inductive assumption. □

The same proof works for a plane and, then, using induction on r, a hyperplane and a Castelnuovo's sequence, we get the following result.

Proposition 4. *Fix positive integers z, k. Let $Z \subset \mathbb{P}^r$, $r \geq 2$, be a general union of z tangent vectors. Then either $h^0(\mathcal{I}_Z(k)) = 0$ or $h^1(\mathcal{I}_Z(k)) = 0$.*

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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