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# OPTIMAL CONVEX COMBINATION BOUNDS OF THE CONTRAHARMONIC AND HARMONIC MEANS FOR THE WEIGHTED GEOMETRIC MEAN

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**Abstract:** We find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha C(a,b) + (1-\alpha)H(a,b) < S(a,b) < \beta C(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $C(a,b)$, $H(a,b)$ and $S(a,b)$ denote the contraharmonic, harmonic, and the weighted geometric means of two positive numbers $a$ and $b$ respectively.

**AMS Subject Classification:** 65N15
**Key Words:** optimal convex combination bound, contraharmonic mean, harmonic mean, the weighted geometric mean

## 1. Introduction

For $a, b > 0$ with $a \neq b$ the weighted geometric mean $S(a,b)$ of $a$ and $b$ with weights $\frac{a}{a+b}$ and $\frac{b}{a+b}$:

$$S \equiv S(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}. \tag{1.1}$$

Let $H(a,b) = \frac{2ab}{a+b}$, $G(a,b) = \sqrt{ab}$, $\mathrm{L}(a,b) = \frac{b-a}{\ln b - \ln a}$,$P(a,b) = \frac{a-b}{4\arctan\sqrt{\frac{a}{b}} - \pi}$,

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$I(a,b)=\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}$, $A(a,b)=\frac{a+b}{2}$, $\mathrm{T}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)}$ and $C(a,b)=\frac{a^2+b^2}{a+b}$ be the harmonic, geometric, logarithmic, Seiffert, identric, arithmetic, centroidal and contraharmonic means of two positive real numbers $a$ and $b$ with $a\neq b$.Then

$$\begin{aligned}&\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<\mathrm{P}(a,b)<I(a,b)<A(a,b)\\&<T(a,b)<S(a,b)<C(a,b)<max(a,b).\end{aligned}\tag{1.2}$$

The weighted geometric mean is a special case of Geni′s mean which can be found in the literature [6] and is related to the identric mean in the literature [9] as follows:

$$\mathrm{S}(a,b)=\frac{\mathrm{I}(a^2,b^2)}{\mathrm{I}(a,b)}.\tag{1.3}$$

For more properties of the mean $S(a,b)$ can be seen in the literature [8], [10] and [14].

In [14], it has been shown that

$$\mathrm{A}_2(\mathrm{a},\mathrm{b})<\mathrm{S}(a,b)\tag{1.4}$$

where

$$\mathrm{A}_2(\mathrm{a},\mathrm{b})=\sqrt{\frac{a^2+b^2}{2}}\tag{1.5}$$

and also that inequality is sharp in a certain sense.

In [15], it has been shown that

$$\mathrm{A}_2(\mathrm{a},\mathrm{b})<\mathrm{S}(\mathrm{a},\mathrm{b})<\sqrt{2}\mathrm{A}_2(\mathrm{a},\mathrm{b})\tag{1.6}$$

holds for all $a,b>0$ with $a\neq b$.

In [16], the authors found the greatest value $p$ and the least value $q$ such that the double inequality

$$\mathrm{M}_p(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)\mathrm{H}^{1-\alpha-\beta}(\mathrm{a},\mathrm{b})<M_q(a,b)\tag{1.7}$$

holds for all $a,b>0$ and $\alpha,\beta>0$ with $a\neq b$.

In [5], the authors found the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha A(a,b)+(1-\alpha)H(a,b)<\mathrm{P}(a,b)<\beta A(a,b)+(1-\beta)H(a,b)\tag{1.8}$$

holds for all $a,b>0$ with $a\neq b$.

The purpose of the present paper is to find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha C(a,b)+(1-\alpha)H(a,b)<S(a,b)<\beta C(a,b)+(1-\beta)H(a,b)\tag{1.9}$$

holds for all $a,b>0$ with $a\neq b$.

## 2. Main Result

**Theorem 2.1.** *The double inequality*

$$\alpha C(a,b) + (1-\alpha)H(a,b) < S(a,b) < \beta C(a,b) + (1-\beta)H(a,b)$$

*holds for all* $a, b > 0$ *with* $a \neq b$. *If and only if* $\alpha \leq \frac{3}{4}$ *and* $\beta \geq 1$.

*Proof.* **Firstly.** We prove that

$$S(a,b) < C(a,b), \tag{2.1}$$

$$S(a,b) > \frac{3}{4}C(a,b) + \frac{1}{4}H(a,b) \tag{2.2}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b > 0$. Let $t = \frac{a}{b} > 1$ and $p \in \{\frac{3}{4}, 1\}$. Then (1.1) leads to

$$y = \frac{S(a,b)}{pC(a,b) + (1-p)H(a,b)} = \frac{t^{\frac{t}{t+1}}(1+t)}{p(t-1)^2 + 2t}, \tag{2.3}$$

$$f(t) = \ln y = \frac{t}{t+1}\mathrm{lnt} + \ln(t+1) - ln\left[pt^2 + (2-2p)t + p\right]. \tag{2.4}$$

Simple computations lead to

$$\lim_{t \to 1+} f(t) = 0, \tag{2.5}$$

$$\lim_{t \to +\infty} f(t) = \ln \lim_{t \to +\infty} \frac{t^{\frac{t}{t+1}}(1+t)}{p(t-1)^2 + 2t} = \ln \frac{1}{p}. \tag{2.6}$$

$$f'(t) = \frac{1}{(t+1)^2} g(t), \tag{2.7}$$

where

$$g(t) = \frac{\left[(2-4p)t^2 + 4p - 2\right] + (pt^2 + 2t - 2pt + p)\mathrm{lnt}}{p(t-1)^2 + 2t}. \tag{2.8}$$

Simple computations lead to

$$\lim_{t\to 1+} g(t) = 0, \tag{2.9}$$

$$\lim_{t\to +\infty} g(t) = +\infty, \tag{2.10}$$

$$g'(t) = \frac{1}{\left[p(t-1)^2 + 2t\right]^2} h(t), \tag{2.11}$$

where

$$h(t) = p^2t^3 + \left(4p^2 - 8p + 4\right)t^2 + (-10p^2 + 4)t + (4p^2 - 8p + 4) + \frac{p^2}{t}. \tag{2.12}$$

$$\lim_{t\to 1+} h(t) = -16p + 12, \tag{2.13}$$

$$\lim_{t\to +\infty} h(t) = +\infty. \tag{2.14}$$

Now we divide the proof into two cases:

**Case 1.** If $p = 1$. Obviously,

$$S(a,b) < C(a,b) \tag{2.15}$$

holds for all $a, b > 0$ with $a \neq b$.

**Case 2.** If $p = \frac{3}{4}$, then from (2.12) we get

$$h'(t) = 3p^2t^2 + 2(4p^2 - 8p + 4)t + (-10p^2 + 4) - \frac{p^2}{t^2}, \tag{2.16}$$

$$\lim_{t\to 1+} h'(t) = 0, \tag{2.17}$$

$$\lim_{t\to+\infty} h'(t) = +\infty, \tag{2.18}$$

$$h''(t) = 6p^2t + (8p^2 - 16p + 8) + \frac{2p^2}{t^3}, \tag{2.19}$$

$$\lim_{t\to1+} h''(t) = 16p^2 - 16p + 8 > 0, \tag{2.20}$$

$$h'''(t) = 6\mathrm{p}^2 - \frac{6\mathrm{p}^2}{\mathrm{t}^4} = 6\mathrm{p}^2(1 - \mathrm{t}^4) > 0. \tag{2.21}$$

From (2.21) we clearly see that $h'''(t) > 0$ for $\forall t \in (1, +\infty)$, hence $h''(t)$ is strictly increasing in $[1, +\infty)$. Because of $h''(1) > 0$, hence $h''(t) > 0$ for $\forall t \in (1, +\infty)$. Therefore, $h'(t)$ is strictly increasing in $[1, +\infty)$. It follows from (2.17) together with the monotonicity of $h'(t)$ that there is $h'(t) > 0$ for $t \in (1, +\infty)$, hence from (2.16) $h(t)$ is strictly decreasing in $[1, +\infty)$. Note that (2.13) becomes

$$\lim_{t\to1+} h(t) = 0 \tag{2.22}$$

for $p = \frac{3}{4}$.

Then it together with the monotonicity of $h(t)$, that $h(t) > 0$ for $t \in (1, +\infty)$, hence from (2.11) $g'(t) > 0$ for $t \in (1, +\infty)$, hence $g(t)$ is strictly increasing in $[1, +\infty)$.

From (2.9) together with the monotonicity of $g(t)$, hence $g(t) > 0$ for $t \in (1, +\infty)$, hence from (2.7) $f'(t) > 0$ for $t \in (1, +\infty)$, hence $f(t)$ is strictly increasing in $[1, +\infty)$.

It follows from (2.5) together with the monotonicity of $f(t)$ that we know $f(t) > 0$ for $t \in (1, +\infty)$, hence from (2.4)

$$y > 1 \tag{2.23}$$

for $t \in (1, +\infty)$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with (2.23).

**Secondly.** We prove that $\beta C(a, b) + (1 - \beta)H(a, b) = C(a, b)$ is the best possible upper convex combination bound of contraharmonic and harmonic means for the weighted geometric mean $S(a, b)$.

In fact, $\forall x > 1$, $\beta \in R$, if $\beta < 1$ we have

$$\lim_{x\to+\infty} \frac{S(x,1)}{\beta C(x,1) + (1-\beta)H(x,1)} = \lim_{x\to+\infty} \frac{x^{\frac{x}{x+1}}(1+x)}{\beta(x-1)^2 + 2x} = \frac{1}{\beta} > 1. \quad (2.24)$$

Inequality (2.24) implies that for $\forall \beta < 1$, $\exists x = X(\beta) > 1$ such that

$$\beta C(x,1) + (1-\beta)H(x,1) < S(x,1)$$

for $\forall x \in (X, +\infty)$.

**Finally.** We prove that $\alpha C(a,b) + (1-\alpha)H(a,b) = \frac{3}{4}C(a,b) + \frac{1}{4}H(a,b)$ is the best possible lower convex combination bound of contra harmonic and harmonic means for the weighted geometric mean $S(a,b)$.

For $\forall t > 1$, $\alpha \in R$, we have

$$y = \frac{S(t,1)}{\alpha C(t,1) + (1-\alpha)H(t,1)} = \frac{t^{\frac{t}{t+1}}(1+t)}{\alpha(t-1)^2 + 2t}. \quad (2.25)$$

Let

$$f(t) = \ln y = \frac{t}{t+1}\text{lnt} + \ln(t+1) - ln\left[\alpha t^2 + (2-2\alpha)t + \alpha\right], \quad (2.26)$$

$$f'(t) = \frac{1}{(t+1)^2} g(t), \quad (2.27)$$

$$g(t) = \frac{\left[(2-4\alpha)t^2 + 4\alpha - 2\right] + (\alpha t^2 + 2t - 2\alpha t + \alpha)\text{lnt}}{\alpha(t-1)^2 + 2t}, \quad (2.28)$$

$$g'(t) = \frac{1}{\left[\alpha(t-1)^2 + 2t\right]^2} h(t), \quad (2.29)$$

$$h(t) = \alpha^2 t^3 + \left(4\alpha^2 - 8\alpha + 4\right)t^2 + (-10\alpha^2 + 4)t + (4\alpha^2 - 8\alpha + 4) + \frac{\alpha^2}{t}. \quad (2.30)$$

It follows from (2.26), (2.28) and (2.30), that

$$f(1) = g(1) = h(1) = 0, \quad (2.31)$$

$$h'(1) = -16\alpha + 12. \tag{2.32}$$

If $\alpha > \frac{3}{4}$, then (2.32) leads to

$$h'(1) < 0. \tag{2.33}$$

From (2.33) and the continuity of $h'(t)$ we see that $\exists \delta = \delta(\alpha) > 0$ such that

$$h'(t) < 0 \tag{2.34}$$

for $\forall \mathrm{t} \in [1, 1 + \delta)$. Then (2.31) and (2.34) imply that

$$f(1) < 0 \tag{2.35}$$

for $\forall \mathrm{t} \in [1, 1 + \delta)$.

That is to say

$$y < 1 \tag{2.36}$$

for $\forall \mathrm{t} \in [1, 1 + \delta)$.

Therefore, $\alpha C(t, 1) + (1 - \alpha) H(t, 1) > S(t, 1)$ for $\mathrm{t} \in (1, 1 + \delta)$ follows from (2.25) and (2.36).

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