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ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF CERTAIN NONLINEAR GENERALIZED α-DIFFERENCE EQUATIONS

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Abstract: In this paper, the authors discuss the asymptotic and oscillatory behavior of solutions of the nonlinear generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}(u(k)+q(k)u(k-\rho)))+t(k)f(u(k-\tau))=0, k\in[a,\infty), \quad (1)$$

where the functions p, q and t are real numbers with $t(k)\geq 0$ and α, ℓ, ρ, τ are positive real. Further, $uf(u)>0$ for $u\neq 0, p(k)>0$ and

$$R(k)=\sum_{r=0}^{\infty}\frac{1}{\alpha^r p(r\ell)}=\infty \quad (2)$$

and $u(k)$ is defined for $k\geq -\max\{\rho,\tau\}$ for all $k\in[a,\infty)$ for some $a\in[0,\infty)$.

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [20]-[22]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{3}$$

no significant progress took place on this line. But recently, E. Thandapani, M.M.S. Manuel, G.B.A.Xavier [5] considered the definition of Δ as given in (3) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined as (3) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [5],[11]-[19]) were obtained. By extending the study related to the sequences of complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike were obtained for difference equation involving Δ_ℓ. The results obtained using Δ_ℓ can be found in ([5]-[19]). Jerzy Popenda and B.Szmanda ([3],[4]) defined Δ as

$$\Delta_\alpha u(k) = u(k+1) - \alpha u(k) \tag{4}$$

and based on this definition, they studied the qualitative properties of a particular difference equation and no one else has handled this operator. In this paper, we have generalized the definition of Δ_α given in (4) and defined and denoted it as

$$\Delta_{\alpha(\ell)} u(k) \ = \ u(k+\ell) - \alpha u(k). \tag{5}$$

where $\alpha > 1$ and $\ell \in [0, \infty)$ and by defining its inverse, several interesting results on number theory were obtained.

In [2], Aleksandra Sternal and Blazej Szmanda obtained sufficient conditions for the asymptotic and oscillatory behaviour of the difference equations are of the form $\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}(u(k)+q(k)u(k-\rho)))+t(k)f(u(k-\tau)) = 0, k \in [a, \infty)$. In this paper the theory is extended from Δ to $\Delta_{\alpha(\ell)}$ for all real $k \in [a, \infty)$ and we discuss the asymptotic and oscillatory behavior of solutions of generalized $\alpha-$difference equation (1).

Throughout this paper, we make use the following assumptions:

(a) $\mathbb{N} = \{0, 1, 2, 3, \dots\},\ \mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,

(b) $\mathbb{N}_\ell(a) = \{a, a+\ell, a+2\ell, \dots\}$.

(c) $\lceil x \rceil$ and $[x]$ denotes the upper integer and the integer part of x respectively.

(d) $j = k - k_i - \left[\frac{k-k_i}{\ell}\right]\ell, k_i \in [0, \infty)$.

(e) $f(u)$ is bounded away from zero, if u is bounded away from zero.

(f) $\sum\limits_{r=0}^{\infty} t(r\ell) = \infty$.

(g) $z(k) = u(k) + q(k)u(k-\rho)$.

2. Preliminaries

In this section, we present some preliminaries which will be useful for future discussion.

Definition 2.1. [14] The inverse of the Generalized α-difference operator denoted by $\Delta^{-1}_{\alpha(\ell)}$ on $u(k)$ is defined as follows. If $\Delta_{\alpha(\ell)} v(k) = u(k)$, then

$$\Delta^{-1}_{\alpha(\ell)} u(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} v(j). \tag{6}$$

where $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right]\ell$.

Lemma 2.2. *[5] If the real valued function $u(k)$ is defined for all $k \in [a, \infty)$ and $\alpha > 1$, then*

$$\Delta^{-1}_{\alpha(\ell)} u(k) = \sum_{r=0}^{\left[\frac{k-a-j-\ell}{\ell}\right]} \frac{u(a+j+r\ell)}{\alpha^{\left\lceil \frac{a+j+\ell-k+r\ell}{\ell} \right\rceil}} + \alpha^{\left\lceil \frac{k-a}{\ell} \right\rceil} u(a+j), \tag{7}$$

for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$.

Definition 2.3. [14] The solution $u(k)$ of (1) is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2+\ell) \le 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory, then it is said to be nonoscillatory (i.e. $u(k)u(k+\ell) > 0$ for all $k \in [k_1, \infty)$).

3. Main Results

In this section we establish conditions for the asymptotic and oscillatory behaviour of solutions of equation (1).

Lemma 3.1. *Assume that there exists a constant* $P_1 < 0$ *such that* $P_1 \leq q(k) \leq 0$.

i) If $u(k)$ *is an eventually positive solution of (1), then* $z(k)$ *and* $p(k)\Delta_{\alpha(\ell)} z(k)$ *are eventually monotonic and either*

$$\lim_{k\to\infty} z(k) = \lim_{k\to\infty} p(k)\Delta_{\alpha(\ell)} z(k) = -\infty \tag{8}$$

or

$$\lim_{k\to\infty} z(k) = \lim_{k\to\infty} p(k)\Delta_{\alpha(\ell)} z(k) = 0, \Delta_{\alpha(\ell)} z(k) > 0 \text{ and } z(k) < 0. \tag{9}$$

In addition, if $P_1 \geq -1$, *then (9) holds and* $u(k)$ *is bounded.*

ii) If $u(k)$ *is an eventually negative solution of (1), then the sequences* $z(k)$ *and* $p(k)\Delta_{\alpha(\ell)} z(k)$ *are monotonic and either*

$$\lim_{k\to\infty} z(k) = \lim_{k\to\infty} p(k)\Delta_{\alpha(\ell)} z(k) = \infty \tag{10}$$

or

$$\lim_{k\to\infty} z(k) = \lim_{k\to\infty} p(k)\Delta_{\alpha(\ell)} z(k) = 0, \Delta_{\alpha(\ell)} z(k) < 0 \text{ and } z(k) > 0. \tag{11}$$

In addition, if $P_1 \geq -1$, *then (11) holds and* $u(k)$ *is bounded.*

Proof. Let $u(k)$ be an eventually positive solution of (1). Then, from (1) there exists a positive integer k_1 such that

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)} z(k)) = -t(k)f(u(k-\tau)) \leq 0 \text{ for } k \geq k_1, \tag{12}$$

that is, $p(k)\Delta_{\alpha(\ell)} z(k)$ is nonincreasing, which implies that $\Delta_{\alpha(\ell)} z(k)$ is eventually of constant sign and in consequence $z(k)$ is monotonic.

First, let there exists $k_2 \geq k_1$ such that $\Delta_{\alpha(\ell)} z(k_2) \leq 0$, then, since $t(k) \neq 0$ eventually, there exists $k_3 > k_2$ such that

$$p(k)\Delta_{\alpha(\ell)} z(k) \leq \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} p(k_3)\Delta_{\alpha(\ell)} z(k_3) = c < 0$$

for $k \geq k_3$.

From Definition 2.1 and Lemma 2.2, we have

$$z(k) \le \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} z(k_3) + c \sum_{r=0}^{\frac{k-k_3-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_3+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_3+j+\ell+r\ell)}$$

$$\to -\infty \text{ as } k \to \infty \tag{13}$$

and hence $z(k) \to -\infty$ as $k \to \infty$. Since $p(k)\Delta_{\alpha(\ell)} z(k)$ is nonincreasing, we have

$$p(k)\Delta_{\alpha(\ell)} z(k) \to L \ge -\infty.$$

If $-\infty < L < 0$, applying Definition 2.1 and Lemma 2.2 to (12), we get $p(k+\ell)\Delta_{\alpha(\ell)} z(k+\ell) = \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} p(k_3+j)\Delta_{\alpha(\ell)} z(k_3+j) -$

$$\sum_{r=0}^{\frac{k-k_3-j}{\ell}} \frac{t(k_3+j+r\ell) f(u(k_3+j+r\ell-\tau))}{\alpha^{\left\lceil \frac{k_3+j-k+r\ell}{\ell} \right\rceil}}$$

and then let $k \to \infty$ to obtain $\displaystyle\sum_{r=0}^{\infty} \frac{t(k_3+j+r\ell) f(u(k_3+j+r\ell-\tau))}{\alpha^{\left\lceil \frac{k_3+j-k+r\ell}{\ell} \right\rceil}}$

$$= \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} p(k_3+j)\Delta_{\alpha(\ell)} z(k_3+j) - L < \infty.$$

The last inequality together with (e) and (f) implies $\liminf_{k\to\infty} u(k) = 0$. Since $z(k)$ is eventually negative, we can choose $k_4 > k_3$ such that $p(k)\Delta_{\alpha(\ell)} z(k) < \frac{L}{2}$ for $k \ge k_4$ and $z(k_4) < 0$. Applying Definition 2.1 and Lemma 2.2 to the above inequality we obtain

$$z(k) < \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} z(k_4) + \frac{L}{2} \sum_{r=0}^{\frac{k-k_4-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_4+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_4+j+\ell+r\ell)}$$

$$< \frac{L}{2} \sum_{r=0}^{\frac{k-k_4-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_4+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_4+j+\ell+r\ell)}, \text{ for } k > k_4.$$

By the assumptions, we have $P_1 u(k-\rho) \le q(k)u(k-\rho) < z(k)$

$$< \frac{L}{2} \sum_{r=0}^{\frac{k-k_4-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_4+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_4+j+\ell+r\ell)}, k > k_4$$

and

$$u(k-\rho) > \frac{L}{2P_1} \sum_{r=0}^{\frac{k-k_4-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_4+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_4+j+\ell+r\ell)} \to \infty,$$

as $k \to \infty$ which contradicts $\liminf_{k\to\infty} u(k) = 0$. Thus, $\lim_{k\to\infty} p(k)\Delta_{\alpha(\ell)} z(k) = -\infty$.

Now, if $\Delta_{\alpha(\ell)} z(k) > 0$ for $k \geq k_1$, then $p(k)\Delta_{\alpha(\ell)} z(k) \to L_1 \geq 0$ as $n \to \infty$. As before, summing (12) from $k \geq k_1$ to m and letting $m \to \infty$ we obtain

$$p(k)\Delta_{\alpha(\ell)} z(k) = L_1 + \sum_{r=0}^{\infty} \frac{t(k+r\ell) f(u(k+r\ell-\tau))}{\alpha^r}$$

which again implies that $\liminf_{n\to\infty} u(k) = 0$.

Suppose that $L_1 > 0$. Then we have, $p(k)\Delta_{\alpha(\ell)} z(k) \geq L_1 > 0$ and a summation shows that $z(k) \to \infty$ as $n \to \infty$. Since $u(k) \geq z(k)$, which leads to $u(k) \to \infty$ as $k \to \infty$, a contradiction. Therefore, $L_1 = 0$. Furthermore, if there exists $k_2 \geq k_1$ such that $z(k_2) \geq 0$, then $\Delta_{\alpha(\ell)} z(k) > 0$ implies that $z(k) \geq z(k_3) > 0$ for all $k \geq k_3$ and some $k_3 > k_2$, which again contradicts $\liminf_{k\to\infty} u(k) = 0$. Therefore, we have $z(k) < 0$ for $k \geq k_1$. Thus $z(k) \to L_2 \leq 0$. If $L_2 < 0$, then

$$P_1 u(k-\rho) \leq u(k) + q(k)u(k-\rho) = z(k) \leq L_2 < 0 \text{ for } k \geq k_1$$

and

$$u(k-\rho) > \frac{L_2}{P_1} > 0, k \geq k_1,$$

which contradicts $\liminf_{n\to\infty} u(k) = 0$. Therefore $L_2 = 0$.

Now we assume that $P_1 \leq -1$. Suppose that (9) does not hold. Then (8) holds, so $z(k) < 0$ for all large k and we have

$$u(k) < -q(k)u(k-\rho) \leq -P_1 u(k-\rho) \leq u(k-\rho)$$

for all large k. But the last inequality implies that $u(k)$ is bounded which contradicts (8). Therefore, (9) holds and $u(k)$ is a bounded solution of (1). The proof of (ii) is similar to that of (i) and hence will be omitted. □

Theorem 3.2. *If there exists a constant P_1 such that*

$$-1 < P_1 \leq q(k) \leq 0, \tag{14}$$

then, every nonoscillatory solution $u(k)$ of (1) tends to 0 as $k \to \infty$.

Proof. If $u(k)$ is an eventually positive solution of (1), then by part i) of Lemma 3.1, we see that $u(k)$ is a bounded solution of (1).

Now, suppose that $\limsup_{k\to\infty} u(k) = a > 0$. Then, there exists a subsequence of $u(k)$, say $u(k_i)$ such that $u(k_i) \to a$ as $i \to \infty$. Then, for all large i, we have

$$0 > z(k_i) \geq u(k_i) + P_1 u(k_i - \rho) \text{ so } u(k_i - \rho) > -\frac{u(k_i)}{P_1}.$$

But this implies that $\lim_{i\to\infty} u(k_i - \rho) \geq -\frac{a}{P_1} > a$, contradicting the choice of a.

Therefore, $u(k) \to 0$ as $k \to \infty$. The proof when $u(k)$ is eventually negative is similar. □

Example 3.3. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\Big(\frac{1}{k}\Delta_{\alpha(\ell)}\Big(\frac{1}{k^2} - \frac{1}{k(k-2\ell)^2}\Big)\Big) = \frac{1}{((k+2\ell)^2)_\ell^{(5)}(k+\ell)_\ell^{(2)}}$$

$$\Big(k(k^2-k-2\ell)(k^2-\ell^2)^2(k-2\ell)^2 - \alpha[(k-\ell)^2-(k+\ell)]k^2(2k+\ell)(k^2-4\ell^2)^2$$

$$+\alpha^2[(k-2\ell)^2-k](k+2\ell)^2(k^2-\ell^2)^2(k+\ell)\Big) \tag{15}$$

where $q(k) = \frac{-1}{k}$, $p(k) = \frac{1}{k}$ and the conditions of Theorem 3.2 hold and ultimately every nonoscillatory solution $u(k)$ of (15) tends to zero as $k \to \infty$. Infact $u(k) = \frac{1}{k^2}$ is one such solution.

The following theorem is an easy consequence of Theorem 3.2.

Theorem 3.4. *If $-1 \leq q(k) \leq 0$, then every unbounded solution of (1) is oscillatory.*

The following theorem shows that, if $q(k)$ is bounded with upper bound less than -1, then all bounded nonoscillatory solutions of (1) tend to zero as $k \to \infty$.

Theorem 3.5. *If there exist constants P_1 and P_2 such that*

$$P_1 \leq q(k) \leq P_2 < -1 \tag{16}$$

then every bounded solution $u(k)$ of (1) is either oscillatory or satisfies $u(k) \to 0$ as $k \to 0$.

Proof. Assume that (1) has a bounded nonoscillatory solution $u(k)$ and let $u(k)$ be eventually positive. By part (i) of Lemma 3.1 either (8) or (9) holds. Clearly (8) cannot hold in view of (16) and the fact that $u(k)$ is bounded. From (9) we have $z(k) < 0$ and $z(k) \to 0$ as $k \to \infty$. Therefore, for any number $\epsilon > 0$ there exists k_1 so that for $k \geq k_1$ we have

$$-\epsilon < z(k) \leq u(k) + P_2 u(k-\rho)$$

or

$$u(k-\rho) < -\frac{u(k)+\epsilon}{P_2}.$$

So

$$u(k) < -\frac{1}{P_2}u(k+\rho) - \frac{1}{P_2}\epsilon \tag{17}$$

and further

$$u(k+\rho) < -\frac{1}{P_2}u(k+2\rho) - \frac{1}{P_2}\epsilon. \tag{18}$$

From (17) and (18) we get

$$u(k) < \Big(-\frac{1}{P_2}\Big)^2 u(k+2\rho) + \Big(-\frac{1}{P_2}\Big)^2\epsilon + \Big(-\frac{1}{P_2}\Big)^2\epsilon.$$

After m iterations, we obtain

$$u(k) < \Big(-\frac{1}{P_2}\Big)^m u(k+m\rho) + \epsilon\sum_{i=1}^{m}\Big(-\frac{1}{P_2}\Big)^i.$$

Let $\lambda = 1 + \dfrac{1}{P_2} > 0$ and $u(k) < M$. Now, choose m large enough so that $\Big(-\dfrac{1}{P_2}\Big)^m < \dfrac{\epsilon}{\lambda M}$. Thus, for every $\epsilon > 0$ there exists $k_2 \geq k_1$ such that for $k \geq k_2$ we have

$$u(k) < \frac{\epsilon}{\lambda} + \epsilon\Big(-\frac{1}{P_2}\Big)\frac{1-\Big(-\frac{1}{P_2}\Big)^m}{1+\frac{1}{P_2}} < 2\frac{\epsilon}{\lambda}.$$

That is, $u(k) \to 0$ as $k \to \infty$.

The proof when $u(k)$ is eventually negative is similar. □

Example 3.6. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\Big(\frac{1}{k}\Delta_{\alpha(\ell)}\Big(\frac{1}{k^2} + \frac{(1-2k)}{k(k-2\ell)^2}\Big)\Big) = \frac{1}{((k+2\ell)^2)_\ell^{(5)}(k+\ell)_\ell^{(2)}}$$

$$\Big((k+\ell)^2k(k^2+(1-2(k+2\ell)))(k+2\ell)+\alpha\ell(k^2+6k\ell+\ell^2-k-\ell)$$

$$(k+2\ell)^2k^2(k-2\ell)^2+\alpha^2[(k-2\ell)^2+(1-2k)k](k+2\ell)^2(k+\ell)^3(k-\ell)^2\Big) \quad (19)$$

where $q(k)=\frac{1-2k}{k}, p(k)=\frac{1}{k}$ all the conditions of Theorem 3.5 hold and ultimately every every bounded solution $u(k)$ of (19) satisfies $u(k)\to 0$ as $k\to\infty$. Infact, $u(k)=\frac{1}{k^2}$ is one such solution.

Theorem 3.7. *If $q(k)$ is eventually nonnegative, then any solution $u(k)$ of (1) is either oscillatory or satisfies $\liminf_{k\to\infty}|u(k)|=0$.*

Proof. Let $u(k)$ be a nonoscillatory solution of (1) and assume that $u(k)$ is eventually positive. Then, as before, (12) implies that $p(k)\Delta_{\alpha(\ell)}z(k)$ is nonincreasing and also $z(k)>0$ eventually, say for $k\geq k_1$. It is easy to see that $\Delta_{\alpha(\ell)}z(k)>0$ for $k\geq k_1$. Indeed, if there exists $k_2\geq k_1$ such that $\Delta_{\alpha(\ell)}z(k_2)\leq 0$, then there exists $k_3\geq k_2$ such that $p(k)\Delta_{\alpha(\ell)}z(k)\leq p(k_3)\Delta_{\alpha(\ell)}z(k_3)=c<0$, since $p(k)\Delta_{\alpha(\ell)}z(k)$ is nonincreasing and $t(k)\equiv 0$ eventually. By (2), we get

$$z(k)\leq \alpha^{\left\lceil\frac{k-k_3}{\ell}\right\rceil}z(k_3)+c\sum_{r=0}^{\frac{k-k_3-\ell-j}{\ell}}\frac{1}{\alpha^{\left\lceil\frac{k_3+j+\ell-k+r\ell}{\ell}\right\rceil}p(k_3+j+\ell+r\ell)}$$

$$\to -\infty \text{ as } k\to\infty,$$

which contradicts that $z(k)>0$ for $k\geq k_1$.

Therefore, $p(k)\Delta_{\alpha(\ell)}z(k)\to L\geq 0$ as $k\to\infty$. Summing (12) from k to $m>n$ with k sufficiently large and then letting $m\to\infty$ we obtain

$$\sum_{r=0}^{\infty}\frac{t(k+r\ell)f(u(k+r\ell-\tau))}{\alpha^{\left\lceil\frac{k+r\ell}{\ell}\right\rceil}}=p(k)\Delta_{\alpha(\ell)}z(k)-L<\infty \quad (20)$$

which, by (e) and (f), implies that $\liminf_{k\to\infty}u(k)=0$. The proof for $u(k)$ eventually negative is similar. □

Example 3.8. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\Big(\frac{1}{k}\Delta_{\alpha(\ell)}\Big(\frac{1}{k^2}+\frac{1}{k(k-2\ell)^2}\Big)\Big)=\frac{1}{((k+2\ell)^2)_\ell^{(5)}(k+\ell)_\ell^{(2)}}$$

$$\Big(k(k^2+k+2\ell)(k^2-\ell^2)^2(k-2\ell)^2-\alpha[(k-\ell)^2+(k+\ell)]k^2(2k+\ell)(k^2-4\ell^2)^2$$

$$+\alpha^2[(k-2\ell)^2-k](k+2\ell)^2(k^2-\ell^2)^2(k+\ell)\Big) \quad (21)$$

where $q(k) = \frac{1}{k}, p(k) = \frac{1}{k}$ and the conditions of Theorem 3.7 hold and ultimately every nonoscillatory solution $u(k)$ of (21) tends to zero as $k \to \infty$. Infact, $u(k) = \frac{1}{k^2}$ is one such solution.

Theorem 3.9. *If $0 \le q(k) \le q, t(k) \ge t > 0$ and there exists a constant $A > 0$ such that $|f(u)| \ge A|u|$ for all u, then all solutions of (1) are oscillatory.*

Proof. Arguing as in the proof of Theorem 3.7 for an eventually positive solution $u(k)$ of (1) we get the equality (20). Further, the assumption, (20) gives

$$Aq \sum_{r=0}^{\infty} \frac{u(k + r\ell - \tau)}{\alpha^{\lceil \frac{k+r\ell}{\ell} \rceil}} \le p(k)\Delta_{\alpha(\ell)} z(k) - L < \infty,$$

which implies that $u(k) \to 0$ as $k \to \infty$ and so $z(k) \to 0$ as $k \to \infty$, which contradicts the fact that $z(k) > 0$ and $\Delta_{\alpha(\ell)} z(k) > 0$. The proof is now complete. □

Example 3.10. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\Big(\frac{1}{k}\Delta_{\alpha(\ell)}\Big((-\alpha)^{\lceil \frac{k}{\ell} \rceil} + \frac{1}{k}(-\alpha)^{\lceil \frac{k-2\ell}{\ell} \rceil}\Big)\Big) = \frac{(-\alpha)^{\lceil \frac{k+2\ell}{\ell} \rceil}}{(k+\ell)k}$$

$$\left(\Big(1 + \frac{\alpha^2}{(k+2\ell)}\Big)k + \Big(1 + \frac{\alpha^2}{(k+\ell)}\Big)(2k+\ell) + \Big(1 + \frac{\alpha^2}{k}\Big)(k+\ell)\right) \qquad (22)$$

where $q(k) = \frac{1}{k}, p(k) = \frac{1}{k}$, all the conditions of Theorem 3.9 hold and hence all solutions $u(k)$ of (22) are oscillatory. Infact $u(k) = (-\alpha)^{\lceil \frac{k}{\ell} \rceil}$ is one such solution.

Theorem 3.11. *Let $q(k) \ge 0$. Then every nonoscillatory solution $u(k)$ of (1) satisfies the following:*

(i) $|u(k)| \le bR(k)$ for some constant $b > 0$ and all large k,

(ii) if $\left(\frac{R(k)}{t(k)}\right)$ is bounded, then $u(k)$ is bounded,

(iii) if $\frac{R(k)}{t(k)} \to 0$ as $k \to \infty$, then $u(k) \to 0$ as $k \to \infty$, where $R(k) = \sum_{r=0}^{[\frac{k-\ell}{\ell}]} \frac{1}{\alpha^r p(r\ell)}$.

Proof. Let $u(k)$ be an eventually positive solution of (1). As before, from (1) we have $\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}z(k)) \leq 0$ for $k \geq k_1$, so summing twice we get

$$z(k) \leq \alpha^{\left\lceil \frac{k-k_3}{\ell} \right\rceil} z(k_1) + p(k_1)\Delta_{\alpha(\ell)}z(k_1)\times$$

$$\times \sum_{r=0}^{\frac{k-k_3-\ell-j}{\ell}} \frac{1}{\alpha^{\left\lceil \frac{k_3+j+\ell-k+r\ell}{\ell} \right\rceil} p(k_3+j+\ell+r\ell)}, k > k_1.$$

By condition (2), we conclude that there is a constant $b > 0$ such that $z(k) \leq bR(k), k \geq k_2 > k_1$. Clearly $u(k) \leq bR(k)$, so (i) holds.

Moreover $q(k)u(k-\rho) \leq bR(k)$ for $k \geq k_2$, and hence (ii) and (iii) follow. The proof when $u(k)$ is eventually negative is similar. □

Example 3.12. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\Big(\frac{1}{k}\Delta_{\alpha(\ell)}\Big(\frac{1}{k^2} - \frac{1}{k(k-2\ell)^2}\Big)\Big) = \frac{1}{((k+2\ell)^2)_\ell^{(5)}(k+\ell)_\ell^{(2)}}$$

$$\Big(k(k^2-k-2\ell)(k^2-\ell^2)^2(k-2\ell)^2 - \alpha[(k-\ell)^2-(k+\ell)]k^2(2k+\ell)(k^2-4\ell^2)^2$$

$$+\alpha^2[(k-2\ell)^2-k](k+2\ell)^2(k^2-\ell^2)^2(k+\ell)\Big) \tag{23}$$

where $q(k) = \frac{-1}{k}, p(k) = \frac{1}{k}$ and the conditions of Theorem 3.2 holds. Infact $u(k) = \frac{1}{k^2}$ is one such solution.

We conclude with an oscillation theorem for (1) in the case $p(k) \equiv 1$ and $q(k) \equiv q > 0$. Now, (1) takes the form

$$\Delta^2_{\alpha(\ell)}(u(k) + q(k)u(k-\rho)) + t(k)f(u(k-\tau)) = 0, k \in [0,\infty). \tag{24}$$

Theorem 3.13. *Suppose that $t(k)$ is k-periodic and f is nondecreasing and satisfies*

$$f(u+v) \leq f(u) + f(v) \text{ if } u,v > 0,$$
$$f(u+v) \geq f(u) + f(v) \text{ if } u,v < 0,$$
$$f(cu) \leq cf(u) \text{ if } c > 0 \text{ and } u > 0$$
$$f(cu) \geq cf(u) \text{ if } c > 0 \text{ and } u < 0.$$

Then every solution of (24) is oscillatory.

Proof. Assume that (24) has a nonoscillatory solution and let $u(k)$ be eventually positive. Then $z(k) = u(k) + qu(k-\rho) > 0$ eventually, say for $k \geq k_1$. From (24) we have $\Delta^2_{\alpha(\ell)} z(k) \leq 0$ for $k \geq k_2 \geq k_1$. We claim that $\Delta_{\alpha(\ell)} z(k) > 0$ for $k \geq k_2$. In fact, if for some $k_3 \geq k_2, \Delta_{\alpha(\ell)} z(k_3) \leq 0$ then since $t(k) \neq 0$ there exists $k_4 > k_3$ such that $\Delta_{\alpha(\ell)} z(k_n) \leq \Delta_{\alpha(\ell)} z(k_4) < 0$ and by summation we see that $z(k) \to -\infty$ as $k \to \infty$. This contradicts the fact that $z(k) > 0$ eventually.

Let $w(k) = z(k) + qz(k-\rho)$. Since from (24) we have $\Delta^2_{\alpha(\ell)} z(k) = -t(k)f(u(k-\tau))$, so, by the assumptions, we get

$$\Delta^2_{\alpha(\ell)} w(k) + q\Delta^2_{\alpha(\ell)} w(k-\rho) + t(k)f(w(k-\tau)) = -t(k)f(u(k-\tau))$$
$$-2qt(n-\rho)f(u(k-\rho-\tau)) - q^2 t(k-2\rho)f(u(k-2\rho-\tau))+$$
$$t(k)f\big(u(k-\tau) + qu(k-\tau-\rho) + q(u(k-\tau-\rho) + qu(k-\tau-\rho))\big)$$
$$\leq -t(k)\big(f(u(k-\tau)) + 2qf(u(k-\tau-\rho)) + q^2 f(u(k-\tau-\rho)\big)$$
$$+t(k)\big(f(u(k-\tau)) + 2qf(u(k-\tau-\rho)) + q^2 f(u(k-\tau-2\rho))\big) = 0.$$

That is

$$\Delta^2_{\alpha(\ell)} w(k) + q\Delta^2_{\alpha(\ell)} w(k-\rho) + t(k)f(w(k-\tau)) \leq 0, \tag{25}$$

and observe that $w(k) > 0$ and $\Delta_{\alpha(\ell)} w(k) > 0$ for $k \geq k_5$, for some $k_5 \geq k_2$. Therefore, $w(k-\tau)$ is increasing for $k \geq k_6$ for some $k_6 \geq k_5$.

Applying Definition 2.1 and Lemma 2.2 to (25) from k_6 to $k-\ell$ we have

$$\Delta_{\alpha(\ell)} w(k) - \alpha^{\left\lceil \frac{k-k_6}{\ell} \right\rceil} \Delta_{\alpha(\ell)} w(k_6) + q\Delta^2_{\alpha(\ell)} w(k-\rho) - q\alpha^{\left\lceil \frac{k-k_6}{\ell} \right\rceil}$$
$$\Delta_{\alpha(\ell)} w(k_6-\rho) + \sum_{r=0}^{\frac{k-k_6-j}{\ell}} \frac{t(k_6+j+r\ell)f(u(k_6+j+r\ell-\tau))}{\alpha^{\left\lceil \frac{k_6+j-k+r\ell}{\ell} \right\rceil}} \leq 0.$$

By the monotonicity of $w(k)$ and f, it follows that

$$f(w(k_6-\tau)) \sum_{r=0}^{\frac{k-k_6-j}{\ell}} \frac{t(k_6+j+r\ell)}{\alpha^{\left\lceil \frac{k_6+j-k+r\ell}{\ell} \right\rceil}}$$
$$\leq \alpha^{\left\lceil \frac{k-k_6}{\ell} \right\rceil} \Delta_{\alpha(\ell)} w(k_6) + p\alpha^{\left\lceil \frac{k-k_6}{\ell} \right\rceil} \Delta_{\alpha(\ell)} w(k_6-\rho),$$

for $k \geq k_6$. Hence, there exists a constant C such that

$$\sum_{r=0}^{\frac{k-k_6-j}{\ell}} \frac{t(k_6+j+r\ell)}{\alpha^{\left\lceil \frac{k_6+j-k+r\ell}{\ell} \right\rceil}} \leq C \text{ for all } k \geq k_6,$$

which contradicts (f). A similar argument can be used in the case of an eventually negative solution. This completes the proof. □

Example 3.14. For the generalized $\alpha-$difference equation

$$\Delta^2_{\alpha(\ell)}\Big((-\alpha)^{\lceil \frac{k}{\ell}\rceil}+\frac{1}{k}(-\alpha)^{\lceil \frac{k-2\ell}{\ell}\rceil}\Big)=\frac{(-\alpha)^{\lceil \frac{k+\ell}{\ell}\rceil}}{(k+\ell)k}\Big(2k^2+2k\ell+2\alpha^2k+\alpha^2\ell\Big) \quad (26)$$

where $q(k)=\frac{1}{k}$, all the conditions of Theorem 3.13 hold and hence all solutions of (26) are oscillatory. Infact $u(k)=(-\alpha)^{\lceil \frac{k}{\ell}\rceil}$ is one such solution.

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