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ON GRADED UNIFORMLY PRIMARY SUBMODULE

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Abstract: Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded uniformly primary submodule and we give a number of results concerning such modules. In fact, our objective is to investigate graded uniformly primary submodules and to examine in particular when graded submodules of a graded module are graded uniformly primary. Special attention has been paid, when graded modules are graded multiplication, to find extra properties of these submodules.

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1. Introduction

Graded primary submodules of graded modules over graded commutative rings have been introduced and studied by various authors, (see, for example [2], [3] ,[5]). Here we introduce the concept of Graded uniformly primary submodules of graded modules over graded commutative rings (see Definition 2) and give a number of its properties.

Before we state some results, let us introduce some notations and terminologies. Let G be a group with identity e and R be a commutative ring.

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Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G). The elements of R_g are called homogeneous of degree g where R_g are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called graded ideal of (R, G) if $I = \bigoplus_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded. For simplicity, we will denote the graded ring (R, G) by R. Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \underset{g\in G}{\oplus} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g\in G}{\cup} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \underset{g\in G}{\oplus} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. For more details, one can look in [4].

Let R be a G-graded ring and M a graded R-module. The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$. A proper graded ideal P of R is said to be graded prime ideal if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$. A proper graded ideal P of R is said to be graded primary ideal if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in Gr(P)$, For more details, one can look in [6]. A proper graded submodule N of a graded R-module M is said to be graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ or $m \in N$. A proper graded submodule N of a graded R-module M is said to be graded primary submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$, so $P = Gr((N :_R M))$ is a graded prime ideal of R, and N is said to be graded P-primary submodule. The graded radical of a graded submodule N of a graded R-module M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$. A graded R-module M is said to be graded finitely generated if there exist $x_{g_1}, x_{g_2}, ..., x_{g_n} \in h(M)$ such that $M = Rx_{g_1} + \cdots + Rx_{g_n}$. For more details, one can look in [1], [2], [3], [5].

2. The Results

The following Lemma is known, but we write it here for the sake of references.

Lemma 1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(i) *If N is a graded submodule of M, $r_g \in h(R)$, $x_h \in h(M)$ and I is a graded ideal of R, then Rx_h, IN and r_gN are graded submodules of M.*

(ii) *If N and K are graded submodules of M, then $N+K$ is graded submodules of M and $(N :_R M)$ is a graded ideal of R.*

Definition 2. Let R be a G-graded ring and M a graded R-module.

(i) A proper graded ideal Q of R is said to be a graded uniformly primary if there exists a positive integer n such that whenever r_g, $t_h \in h(R)$ with $r_g t_h \in Q$, then either $r_g \in Q$ or $t_h^n \in Q$. We say that a graded uniformly primary ideal Q has order s (and write $ord_R(Q) = s$), if s is the smallest positive integer for which the aforementioned property holds.

(ii) A proper graded submodule N of M is said to be a graded uniformly primary if there exists a positive integer n such that whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $r_g m_h \in N$, then either $m_h \in N$ or $r_g^n \in (N :_R M)$. We say that a graded uniformly primary submodule N has order s (and write $ord_R(N) = s$), if s is the smallest positive integer for which the aforementioned property holds.

Theorem 3. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then N is a graded uniformly P-primary submodule of M if and only if the following two conditions hold:*

(i) *N is a graded P-primary submodule of M, and*

(ii) *there exists a positive integer n such that $P = \{r \in R : r_g^n \in (N :_R M)$ for all $g \in G\}$. Moreover $ord_R(N) = s$ if and only if s is the smallest positive integer for which condition (ii) holds.*

Proof. (⇒) Let N be a graded uniformly P-primary submodule of M. Then condition (i) is trivially satisfied. Let $r \in P$ and $g \in G$. Then there are a positive integer n_g and $m_h \in h(M)$ such that $r_g^{n_g} m_h = r_g^{n_g-1} r_g m_h \in N$, but $r_g^{n_g-1} m_h \notin N$. Since $ord_R(N) = s$, we have $r_g^s \in (N :_R M)$, and so condition (ii) is established.
(⇐) Suppose condition (i) and (ii). Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N$ and $m_h \notin N$. Since N is a graded P-primary, we have $r_g \in P$ and condition (ii) provides for positive integer n, independent of r_g, such that $r_g^n \in (N :_R M)$. Thus N is a graded uniformly primary submodule of M. Finally, by the above consideration, $ord_R(N) = s$ if and only if s is the smallest positive integer for which condition (ii) holds. □

Theorem 4. *Let R be a G-graded ring and Q a graded ideal of R. Then Q is a graded uniformly P-primary ideal of R if and only if the following two conditions hold:*

(i) *Q is a graded P-primary ideal of R, and*

(ii) *There exists a positive integer n such that $P = \{r \in R : r_g^n \in Q$ for all $g \in G\}$. Moreover $ord_R(Q) = s$ if and only if s is the smallest positive integer for which condition (ii) holds.*

Proof. The proof is similar to that of Theorem 3. □

Theorem 5. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If there exists a positive integer s such that for any graded ideal I of R and graded submodule V of M, $IV \subseteq N$ implies $V \subseteq N$ or $I^s \subseteq (N :_R M)$, then N is a graded uniformly primary submodule with $ord_R(N) \leq s$.*

Proof. Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N$. So we have $(r_g)(m_h) \subseteq N$ where (r_g) is a graded ideal of R and (m_h) is a graded submodule of M. By our assumption we obtain $(m_h) \subseteq N$ or $(r_g)^s \subseteq (N :_R M)$ which implies that $m_h \in N$ or $r_g^s \subseteq (N :_R M)$. Thus N is a graded uniformly primary submodule with $ord_R(N) \leq s$. □

Lemma 6. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded uniformly primary submodule of M with order s, then for every $m_\lambda \in h(M) \backslash N$, $(N :_R m_\lambda)$ is a graded uniformly primary ideal of R and $Gr((N :_R m_\lambda)) = Gr((N :_R M))$.*

Proof. Suppose that N is a graded uniformly primary submodule of M with order s. Let $m_\lambda \in h(M)\backslash N$ and $r_g, t_h \in h(R)$ such that $r_g t_h \in (N :_R m_\lambda)$ and $t_h \notin (N :_R m_\lambda)$. Hence $r_g t_h m_\lambda \in N$ and $t_h m_\lambda \notin N$. Since N is a graded uniformly primary submodule of M with order s, $r_g^s \in (N :_R M)$, it follows that $r_g^s \in (N :_R m_\lambda)$. Thus $(N :_R m_\lambda)$ is a graded uniformly primary ideal of R. It is clear that $Gr(N :_R M) \subseteq Gr((N :_R m_\lambda))$. Now we will prove the reverse inclusion. Let $k \in Gr((N :_R m_\lambda))$ and $g \in G$. Then $k_g^s m_\lambda \in N$. By assumption there exists $l \in \mathbb{Z}^+$ such that $(k_g^s)^l \in (N :_R M)$. So $k_g \in Gr(N :_R M)$. Thus $Gr((N :_R m_\lambda)) \subseteq Gr(N :_R M)$. □

Theorem 7. *Let R be a G-graded ring, M a graded finitely generated R-module and N a proper graded submodule of M. Then the following statements are equivalent.*

(i) *N is a graded uniformly primary submodule of M.*

(ii) *For $m_g \in h(M)\backslash N$, $(N :_R m_g)$ is a graded uniformly primary ideal of R and $Gr((N :_R m_g)) = Gr((N :_R M))$.*

Proof. $(i) \Rightarrow (ii)$ See Lemma 6.
$(ii) \Rightarrow (i)$ Since M is a graded finitely generated, there exist $m_{g_1}, m_{g_2}, \dots, m_{g_n} \in h(M)$ such that $M = Rm_{g_1} + Rm_{g_2} + \cdots + Rm_{g_n}$. Let $s_i = ord_R((N :_R m_{g_i}))$.Without loss of generality, we may assume that $m_{gi} \notin N$. Now, let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N$ and $m_h \notin N$. Hence $r_g \in (N :_R m_h) \subseteq Gr((N :_R m_h)) = Gr((N :_R M)) = Gr((N :_R m_{g_i}))$. By Theorem 4, $r_g^{s_i} \in (N :_R m_{g_i})$. if $l = max\{ord_R(((N :_R m_{g_i})) : i = 1, 2, \dots n\}$, then $r_g^l \in (N :_R M)$. Thus N is a graded uniformly primary submodule of M. □

Lemma 8. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded uniformly primary submodule of M, then $(N :_R M)$ is a graded uniformly primary ideal of R.*

Proof. Assume that N is a graded uniformly primary submodule of M of order s and let r_g, $t_h \in h(R)$ such that $r_g\ t_h \in (N :_R M)$ and $r_g \notin (N :_R M)$. Then there exists $m_\lambda \in h(M)\backslash N$ such that $r_g m_\lambda \notin N$ and $r_g t_h m_\lambda \in N$. Since N is a graded uniformly primary submodule of M of order s, we have $t_h^s \in (N :_R M)$. Hence $(N :_R M)$ is a graded uniformly primary ideal of R. □

Recall that a graded R-module M is called graded multiplication if for each graded submodule N of M; $N = IM$ for some graded ideal I of R. One can easily show that If N is graded submodule of a graded multiplication module M, then $N = (N :_R M)M$, (see, [5, Definition 2]).

Theorem 9. *Let R be a G-graded ring, M a graded multiplication R-module and N a graded submodule of M. Then N is a graded uniformly primary submodule of M if and only if $(N :_R M)$ is a graded uniformly primary ideal of R.*

Proof. ($\Rightarrow$) See Lemma 8.
($\Leftarrow$) Assume that $(N :_R M)$ is a graded uniformly primary ideal of R of order s and let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N$ and $m_h \notin N$. Since M is a graded multiplication, we have $Rm_h = JM$ for some graded ideal J of R. Hence $r_g JM \subseteq N$, i.e., $r_g J \subseteq (N :_R M)$. Since $(N :_R M)$ is a graded uniformly primary ideal of R of order s and $J \nsubseteq (N :_R M)$, we conclude that $r_g^s \in (N :_R M)$. Thus N is a graded uniformly primary submodule of M. □

Lemma 10. *[3, Theorem 2.2] Let R be a G-graded ring, M a faithful graded multiplication R-module and Q a graded primary ideal of R. If $r_g m_h \in QM$ for $r_g \in h(R)$, $m_h \in h(M)$, then $r_g \in Gr(Q)$ or $m_h \in QM$.*

Theorem 11. *Let R be a G-graded ring, M a faithful graded multiplication R-module and Q a graded uniformly P-primary ideal of R of order s such that $QM \neq M$. Then QM is a graded uniformly primary submodule of M of order at most s.*

Proof. Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in QM$ and $m_h \notin QM$. By Lemma 10 and Theorem 4 $r_g \in Gr(Q) = \{r \in R : r_g^s \in Q$ for all $g \in G\}$. Hence $r_g^s \in Q \subseteq (QM :_R M)$. Therefore QM is a graded uniformly primary submodule of M of order at most s. □

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