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# SOLVING STIFF DIFFERENTIAL EQUATIONS USING A-STABLE BLOCK METHOD

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**Abstract:** This paper will present the two-point block one-step method for solving stiff ordinary differential equations (ODE s). The propose block method is A-stable and the order is three. The solutions will be obtain simultaneously in block and produces two approximate solutions using constant step size. The method is similar as the one-step method and it is self-starting but the implementation is based on the predictor and corrector formulae. The order of the method will be discussed. The numerical results is presented to illustrate the applicability of the propose method. The results clearly shown that the propose method is able to produce comparable and better results compared to the existing method when solving stiff differential equations.

**AMS Subject Classification:** 65L06, 65L10
**Key Words:** stiff ordinary differential equation, block method, two-point

## 1. Introduction

Stiff equation in mathematics is a differential equation that may gives unstable result if it is solve using certain numerical method, unless the step size taken

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is very small. Nowadays, there are many methods that has been proposed to solve stiff problems such as in Musa *et al.* [1], Ababneh *et al.* [9] and Nasir *et al.* [6]. Musa *et al.*[1] had derived a new block extended backward differential formula suitable for the integration of stiff initial value problem by improving the existing block backward differentiation formula. Ababneh *et al.* [9] had introduced the modified explicit third order Runge-Kutta method based on contraharmonic mean (C0M) that allows reducing the stiffness in some sense. Sharmila *et al.* [10] had developed a new third order weight Runge-Kutta formula based on Centoridal Mean (CeM) for solving stiff equation. Nasir *et al.* [6] had developed a two-point implicit code in the form of fifth order block backward differential formula (BBDF5) for solving first order stiff ODEs using constant step size. While Yatim *et al.*[13] had proposed the block backward differentiation formula (BBDF) using varies of order and step size.
Rosser [4] has proposed the method of two point block one-step method based on Newton-Cotes type. In 2003, Majid *et al.*[15] has improved this method by proposed a block method by integrating using the closest point in the interval based on Newton backward divided difference formula but the author has solved the non stiff ODEs where the implementation only used the fixed iteration or known as simple iteration. In this paper, we propose the two-point block one-step method in Majid *et al.*[15] with Newton's iteration for solving stiff differential equation. The order, consistency, zero stable and stability of the method will be discussed in this paper.
The following general form of initial value problem (IVP) for first order stiff ODEs will be consider:

$$y' = f(x, y), \qquad y(a) = y_0 \qquad a \leq x \leq b \tag{1}$$

where $a$ and $b$ are finite.

## 2. Derivation of the Two-Point Block One-Step Method

In Figure 1, the interval $[a, b]$ is divided into a series of blocks which each block containing two points with the step size $2h$. The value of the two point, $y_{n+1}$ and $y_{n+2}$ will be approximate simultaneously at the point $x_{n+1}$ and $x_{n+2}$. The value of $x_n$ will be used for the starting of $k$ block and the last point in $k$ block i.e. $x_{n+2}$ will be used for the starting of $k+1$ block and this process will continue till the end of the series of block.
The following is the derivation of the formulae in Majid *et. al.*[15]:

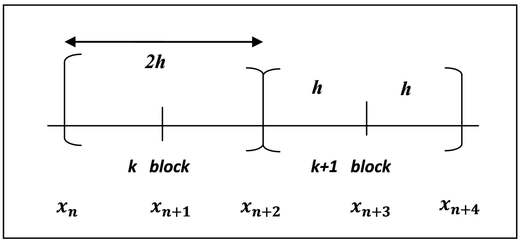


Figure 1: Two-point block one-step method

Let $x_{n+1} = x_n + h$,

$$\int_{x_{\mathrm{n}}}^{x_{\mathrm{n+1}}} y' dx = \int_{x_{\mathrm{n}}}^{x_{\mathrm{n+1}}} f(x,y)dx$$

or

$$y(x_{n+1}) = y(x_n) + \int_{x_{\mathrm{n}}}^{x_{\mathrm{n+1}}} f(x,y)dx \tag{2}$$

Then, $f(x,y)$ in Eq. (2) will be replace with polynomial interpolation and give the following equation,

$$P_{n+2}(x) = \sum_{m=0}^{k} (-1)^m \begin{pmatrix} -s \\ m \end{pmatrix} \nabla^m f_{n+2} \text{ where } s = \frac{x - x_{n+2}}{h}.$$

By changing the limit of integration and replace $dx = hds$, we will obtain

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + h \sum_{m=0}^{k} \gamma^m \nabla^m f_{n+2} \\ \text{where } \gamma^m &= (-1)^m \int_{-2}^{-1} \begin{pmatrix} -s \\ m \end{pmatrix} ds. \end{aligned} \tag{3}$$

and solve Eq. (3), hence we will obtain the first formula of the two-point block one-step method as follows,

$$y_{n+1} = y_n + \frac{h}{12}\left(5f_n + 8f_{n+1} - f_{n+2}\right). \tag{4}$$

For the second point, taking $x_{n+2} = x_{n+1} + h$ and integrate $f$ from $x_{n+1}$ to $x_{n+2}$, changing the limit of integration and replace $dx = hds$ in Eq. (2) gives

$$y\left(x_{n+2}\right) = y\left(x_{n+1}\right) + h \sum_{m=0}^{k} \delta_m \nabla^m f_{n+2}$$

$$\text{where } \delta = (-1)^m \int_{-1}^{0} \binom{-s}{m} ds \, . \tag{5}$$

Then, solve Eq. (5) and obtain the second formula of the two-point block one-step method as follows,

$$y_{n+2} = y_{n+1} + \frac{h}{12}\left(-f_n + 8f_{n+1} + 5f_{n+2}\right). \tag{6}$$

The two-point block one-step method in Eq. (4) and (6) can be written in a matrix difference equation as follows:

$$A^{(0)}Y_m = A^{(1)}Y_{m-1} + h(B^{(0)}F_m + B^{(1)}F_{m-1}) \tag{7}$$

where

$$A^{(0)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} \frac{8}{12} & \frac{-1}{12} \\ \frac{8}{12} & \frac{5}{12} \end{bmatrix} \quad B^{(1)} = \begin{bmatrix} 0 & \frac{5}{12} \\ 0 & \frac{-1}{12} \end{bmatrix}$$

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} F_{m-1} = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

or also can be written as

$$\alpha Y_m = \beta Y'_m \tag{8}$$

The order of this developed method is identified by referring to Fatunla [14]. By applying the formulae for the constants $C_p$, the formulae is defined as

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=0}^{k} j\alpha_j - \beta_j, \\
C_2 &= \sum_{j=0}^{k} \frac{j^2\alpha_j}{2!} - j\beta_j, \\
C_3 &= \sum_{j=0}^{k} \frac{j^3\alpha_j}{3!} - \frac{j^2\beta_j}{2!}, \\
&\vdots
\end{aligned}$$

$$C_p = \sum_{j=0}^{k} \frac{j^p\alpha_j}{p!} - \frac{j^{p-1}\beta_j}{p-1!}. \text{ where } p = 4, 5, 6, ... \tag{9}$$

Therefore, the order and error constant of the two point block one-step method will be compute by using Eq. (9) and the following steps will be obtained:
For $p = 0$,

$$C_0 = \sum_{j=0}^{k} \alpha_j = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $p = 1$,

$$C_1 = \sum_{j=0}^{k} j\alpha_j - \beta_j = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{5}{12} \\ \frac{-1}{12} \end{pmatrix} - \begin{pmatrix} \frac{8}{12} \\ \frac{8}{12} \end{pmatrix} - \begin{pmatrix} \frac{-1}{12} \\ \frac{5}{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $p = 2$,

$$C_2 = \sum_{j=0}^{k} \frac{j^2\alpha_j}{2!} - j\beta_j = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{4}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{8}{12} \\ \frac{8}{12} \end{pmatrix} - 2\begin{pmatrix} \frac{-1}{12} \\ \frac{5}{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $p = 3$,

$$C_3 = \sum_{j=0}^{k} \frac{j^3\alpha_j}{3!} - \frac{j^2\beta_j}{2!} = \frac{1}{6}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{8}{6}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{8}{12} \\ \frac{8}{12} \end{pmatrix} - \frac{4}{2}\begin{pmatrix} \frac{-1}{12} \\ \frac{5}{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $p = 4$,

$$\begin{aligned} C_4 &= \sum_{j=0}^{k} \frac{j^4\alpha_j}{4!} - \frac{j^3\beta_j}{3!} = \frac{1}{24}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{16}{24}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} \frac{8}{12} \\ \frac{8}{12} \end{pmatrix} - \frac{8}{6}\begin{pmatrix} \frac{-1}{12} \\ \frac{5}{12} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{24} \\ \frac{-1}{24} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

The method is order p if $C_0 = C_1 = \cdots = C_p = 0$ and $C_(p+1) \neq 0$ is the error constant. Thus, by implementing Eq. (9) the corrector formulae in (4) and (6) is of order three and the error constant is

$$C_{p+1} = C_4 = \begin{pmatrix} \frac{1}{24} \\ -\frac{1}{24} \end{pmatrix}$$

**Definition 1.** The method is zero stable provided the roots $R_j, j = 1(1)k$ of the first characteristic polynomial $\rho(r)$ specified as

$$\rho\left(r\right) = \det\left[\sum_{i=0}^{k} A^{(i)} R^{k-i}\right] = 0, \qquad A^{(0)} = -I$$

satisfies $|R_j| \leq 1$, and for those roots with $|R_j| = 1$, the multiplicity must not exceed 2.

By referring to Eq. (7), we have the first characteristics polynomial of the two-point block one-step method given as follows:

$$\rho\left(r\right) = \det\left[RA^{(0)} - A^{(1)}\right] = 0$$

Hence,

$$\begin{aligned} \det\left[R\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] &= 0 \\ \det\begin{bmatrix} R & -1 \\ R & R \end{bmatrix} &= R^2 + R = 0 \\ &= R\left(R+1\right) = 0. \end{aligned}$$

Therefore

$$R_1 = 0, R_2 = -1$$

Therefore, from the definition of zero stable, two-point block one-step method is zero stable since $|R_j| \leq 1$. Since this method is order 3, it has prove that this method is consistent and also convergence because of its consistency and zero stable.

## 3. Stability Region

In this section, we will discuss the stability of two-point block one-step method on a linear first order problem when the method is applied to the test equation:

$$y' = f = \lambda y \tag{10}$$

The Eq. (4) and (6) can be written in matrix form as

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} \frac{8}{12} & -\frac{1}{12} \\ \frac{8}{12} & \frac{5}{12} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$

$$+h\begin{bmatrix} 0 & \frac{5}{12} \\ 0 & -\frac{1}{12} \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} \tag{11}$$

Then, we substitute Eq. (10) into (11) we get

$$\begin{bmatrix} 1-\frac{8}{12}h\lambda & \frac{1}{12}h\lambda \\ -1-\frac{8}{12}h\lambda & 1-\frac{5}{12}h\lambda \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}$$
$$+h\begin{bmatrix} 0 & \frac{5}{12}\lambda \\ 0 & -\frac{1}{12}\lambda \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \tag{12}$$

where

$$A=\begin{bmatrix} 1-\frac{8}{12}h\lambda & \frac{1}{12}h\lambda \\ -1-\frac{8}{12}h\lambda & 1-\frac{5}{12}h\lambda \end{bmatrix},\ B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C=\begin{bmatrix} 0 & \frac{5}{12}\lambda \\ 0 & -\frac{1}{12}\lambda \end{bmatrix}. \tag{13}$$

**Definition 2.** A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane Re $h\lambda < 0$.

The method is A-stable by evaluating the $det|At-(B+Ch)|=0$ to obtain the characteristic polynomial as follows:

$$\left(-h+1+\frac{1}{3}h^2\right)t^2+\left(-\frac{1}{3}h^2-1-h\right)t=0 \tag{14}$$

and the stability region is shown in Figure 2.
Figure 2 shows the stability region covers the left half-plane and this indicates that the method is A-stable. This proves that the method is suitable for solving stiff differential equations.

## 4. Implementation of the Method

In this research, we will implement the Newton's iteration together with the block method when solving the stiff problems. The following is the description of the implementation in the method. Firstly, we discuss the definition of error:

**Definition 3.** Let $y_i$ and $y(x_i)$ be the approximate and exact solutions of (1) respectively, then given the absolute error as follows

$$(error_i)_t = \left|(y_i)_t - (y(x_i))_t\right| \tag{15}$$

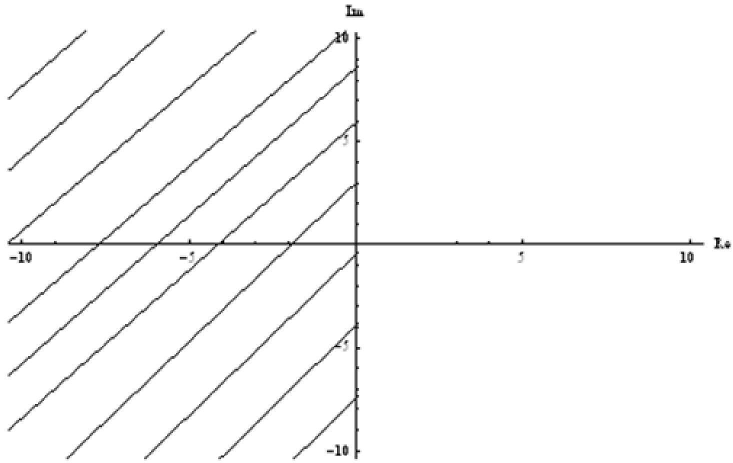


Figure 2: Stability region of two-point block one-step method

The maximum error is given by

$$MAXE = \max_{1 \le i \le T} ( \max_{1 \le i \le N} (error_i)_t).$$

where $T$ is the total step and $N$ is the number of equations.
Let $y_{n+1}^{(i+1)}$ denote the $(i+1)^{th}$ iterate and

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)} \qquad\qquad j = 1, 2. \tag{16}$$

Define

$$\begin{aligned} F_1 &= y_{n+1} - y_n - \frac{h}{12}(5f_n + 8f_{n+1} - f_{n+2}) \\ F_2 &= y_{n+2} - y_{n+1} - \frac{h}{12}\left(-f_n + 8f_{n+1} + 5f_{n+2}\right). \end{aligned} \tag{17}$$

By definition of Newton's iteration, we take Eq. (17) into the form

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \frac{\left[F(y_{n+j}^{(i)})\right]}{\left[F'(y_{n+j}^{(i)})\right]} \tag{18}$$

which is equivalent to

$$e_{n+j}^{(i+1)} = -\frac{\left[F(y_{n+j}^{(i)})\right]}{\left[F'(y_{n+j}^{(i)})\right]} \tag{19}$$

or

$$\left[F'(y_{n+j}^{(i)})\right] e_{n+j}^{(i+1)} = -\left[F(y_{n+j}^{(i)})\right] \tag{20}$$

Eq. (20) can be rewritten as

$$\begin{pmatrix} -1-\frac{5}{12}h\frac{\partial f_n}{\partial y_n} & 1-\frac{8}{12}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{1}{12}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \\ \frac{1}{12}h\frac{\partial f_n}{\partial y_n} & -1-\frac{8}{12}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1-\frac{5}{12}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix} \begin{pmatrix} e_n^{(i+1)} \\ e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_n^{(i)} \\ y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \end{pmatrix} + h \begin{pmatrix} \frac{5}{12} & \frac{8}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{8}{12} & \frac{5}{12} \end{pmatrix} \begin{pmatrix} f_n^{(i)} \\ f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \end{pmatrix} \tag{21}$$

In this research, we will use $(error_i)_t \leq 10^{-6}$ as the error bound of Newton's iteration.

## 5. Result and Discussion

There are five tested problems will be test using this block method and compare it with existing method. The algorithm was written in C language.

Problem 1:
$y' = -10xy \qquad y(0) = 1 \qquad 0 \leq x \leq 10$
Exact solution: $y(x) = e^{-5x^2}$
Source: Musa *et al.* [1]

Problem 2:
$y_1' = -100y_1 + 9.901y_2 \qquad y_1(0) = 1 \qquad 0 \leq x \leq 10$
$y_2' = 0.1y_1 - y_2 \qquad y_2(0) = 10$
Exact solution:
$y_1(x) = e^{-0.99x}$
$y_2(x) = 10e^{-0.99x}$
Eigenvalues -0.99 and -100.01
Source: Musa *et al.* [1]

Problem 3:
$y_1' = 9y_1 + 24y_2 + 5\cos x - \frac{1}{3}\sin x \qquad y_1(0) = \frac{4}{3} \qquad 0 \leq x \leq 10$
$y_2' = -24y_1 - 51y_2 - 9\cos x + \frac{1}{3}\sin x \qquad y_2(0) = \frac{2}{3}$

Exact solution:
$y_1(x) = 2e^{-3x} - e^{-39x} + \frac{1}{3}\cos x$
$y_2(x) = -e^{-3x} + 2e^{-39x} - \frac{1}{3}\cos x$
Source: Musa *et al.* [1]

Problem 4:
$y' = -100(y - x^3) + 3x^2 \qquad y(0) = 0 \qquad 0 \le x \le 10$
The eigenvalue is $\lambda = -100$
Exact solution:
$y(x) = x^3$
Source: Nasir *et al.* [6]

Problem 5:
$y_1' = -2y_1 + y_2 + 2\sin x \qquad y_1(0) = 2 \qquad 0 \le x \le 10$
$y_2' = 998y_1 - 999y_2 + 999(\cos x - \sin x) \qquad y_2(0) = 3$
The eigenvalue is $\lambda_1 = -1$ and $\lambda_2 = -1000$
Exact solution:
$y_1(x) = 2e^{-x} + \sin x$
$y_2(x) = 2e^{-x} + \cos x$
Source: Nasir *et al.* [6]

Notations used in the following tables are:

| | |
|---|---|
| 2PBOSM | Two-point block one-step method proposed in this research |
| 3BEBDF | 3-point block extended backward differential formula proposed by Musa *et al.* [1] |
| BBDF(5) | Fifth order block backward differentiation formula proposed by Nasir *et al.* [6] |
| MAXE | Maximum error |
| h | Step size |
| AVE | Average of error |
| TS | Total step |
| TIME | Time in microseconds |
| TF | Total function |
| - | No data has been reported in the reference |

Table 1: Comparison between 2PBOSM and 3BEBDF for problem 1

| h | 3BEBDF | | 2PBOSM | | | | |
|---|---|---|---|---|---|---|---|
| | MAXE | TIME | TS | TF | MAXE | AVGE | TIME |
| $10^{-1}$ | - | - | 50 | 388 | 9.37 (-4) | 2.57(-5) | 143 |
| $10^{-2}$ | 1.24(-2) | 3022 | 500 | 3540 | 1.25 (-7) | 3.54 (-9) | 928 |
| $10^{-3}$ | 7.36(-4) | 20867 | 5000 | 32832 | 4.58 (-8) | 2.32(-9) | 7899 |
| $10^{-4}$ | 7.07(-5) | 208115 | 50000 | 322768 | 4.55 (-8) | 1.00(-9) | 45962 |
| $10^{-5}$ | 7.04(-6) | 2068090 | 500000 | 3174282 | 4.65 (-8) | 1.23(-9) | 279034 |
| $10^{-6}$ | 7.03(-7) | 20760900 | 5000000 | 30421236 | 5.80 (-8) | 2.86(-9) | 2410929 |

Table 2: Comparison between 2PBOSM and 3BEBDF for problem 2

| h | 3BEBDF | | 2PBOSM | | | | |
|---|---|---|---|---|---|---|---|
| | MAXE | TIME | TS | TF | MAXE | AVGE | TIME |
| $10^{-1}$ | - | - | 50 | 1656 | 1.25(-4) | 2.81(-5) | 456 |
| $10^{-2}$ | 8.35(-2) | 4188 | 500 | 9384 | 2.56(-3) | 3.98(-4) | 2117 |
| $10^{-3}$ | 9.10(-3) | 40190 | 5000 | 85544 | 2.55(-4) | 3.89(-5) | 18134 |
| $10^{-4}$ | 9.18(-4) | 398665 | 50000 | 763616 | 2.55(-5) | 3.92(-6) | 79488 |
| $10^{-5}$ | 9.19(-5) | 3969950 | 500000 | 6708220 | 2.55(-6) | 4.55(-6) | 414915 |
| $10^{-6}$ | 9.19(-6) | 39750100 | 5000000 | 60000000 | 6.92(-7) | 7.87(-8) | 3342770 |

Table 3: Comparison between 2PBOSM and 3BEBDF for problem 3

| h | 3BEBDF | | 2PBOSM | | | | |
|---|---|---|---|---|---|---|---|
| | MAXE | TIME | TS | TF | MAXE | AVGE | TIME |
| $10^{-2}$ | 1.68(-1) | 13784 | 500 | 24936 | 1.09(-3) | 4.66(-6) | 4813 |
| $10^{-3}$ | 5.15(-2) | 136043 | 5000 | 116288 | 1.61(-4) | 4.65(-5) | 46762 |
| $10^{-4}$ | 6.96(-3) | 1356710 | 50000 | 987712 | 2.68(-4) | 1.56(-5) | 214751 |
| $10^{-5}$ | 7.17(-4) | 13584800 | 500000 | 8702740 | 3.17(-5) | 2.67(-6) | 1698730 |
| $10^{-6}$ | 7.19(-5) | 135862000 | 5000000 | 60402656 | 5.59(-6) | 1.38(-6) | 11440033 |

Table 4: Comparison between 2PBOSM and BBDF(5) for problem 4

| h | BBDF(5) | | | 2PBOSM | | |
|---|---|---|---|---|---|---|
| | TS | MAXE | AVGE | TS | MAXE | AVGE |
| $10^{-1}$ | - | - | - | 50 | 2.41(-7) | 7.67(-8) |
| $10^{-2}$ | - | - | - | 500 | 4.00(-7) | 6.14(-9) |
| $10^{-3}$ | - | - | - | 5000 | 6.07(-8) | 5.40(-9) |
| $10^{-4}$ | 50000 | 1.20(-4) | 2.00(-5) | 50000 | 9.21(-8) | 2.93(-9) |
| $10^{-6}$ | 500000 | 1.20(-6) | 2.00(-7) | 500000 | 1.16(-7) | 3.43(-9) |
| $10^{-8}$ | 50000000 | 5.48(-8) | 8.54(-9) | 50000000 | 2.48(-7) | 6.79(-8) |

Table 5: Comparison between 2PBOSM and BBDF(5) for problem 5

| h | BBDF(5) | | | 2PBOSM | | |
|---|---|---|---|---|---|---|
| | TS | MAXE | AVGE | TS | MAXE | AVGE |
| $10^{-1}$ | - | - | - | 50 | 2.65(-2) | 3.17(-3) |
| $10^{-2}$ | - | - | - | 500 | 1.96(-3) | 5.55(-3) |
| $10^{-3}$ | - | - | - | 5000 | 2.08(-4) | 6.44(-5) |
| $10^{-4}$ | 50000 | 1.03(-4) | 1.34(-5) | 50000 | 1.88(-5) | 6.98(-6) |
| $10^{-6}$ | 500000 | 1.03(-6) | 1.34(-7) | 500000 | 1.01(-6) | 4.70(-7) |
| $10^{-8}$ | 50000000 | 6.78(-9) | 5.11(-10) | 50000000 | 6.37(-8) | 1.13(-8) |

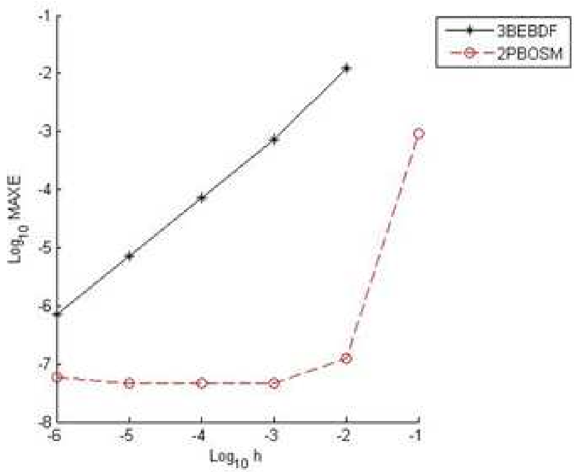


Figure 3: Graph maximum errors versus $h$ for 3BEBDF and 2PBOSM when solving problem 1

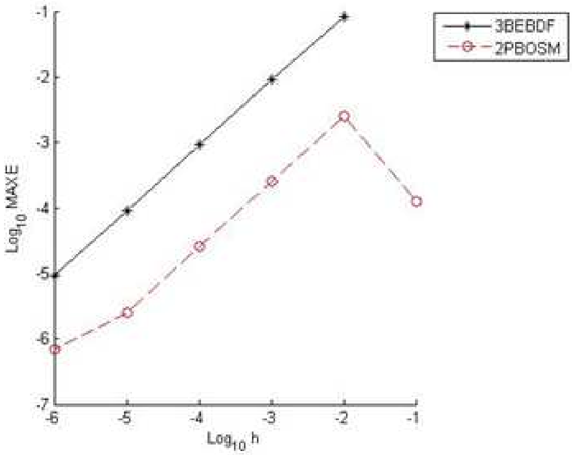


Figure 4: Graph maximum errors versus $h$ for 3BEBDF and 2PBOSM when solving problem 2

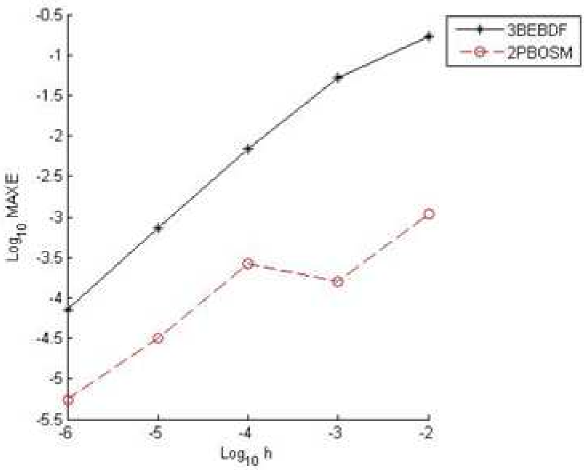


Figure 5: Graph maximum errors versus $h$ for 3BEBDF and 2PBOSM when solving problem 3

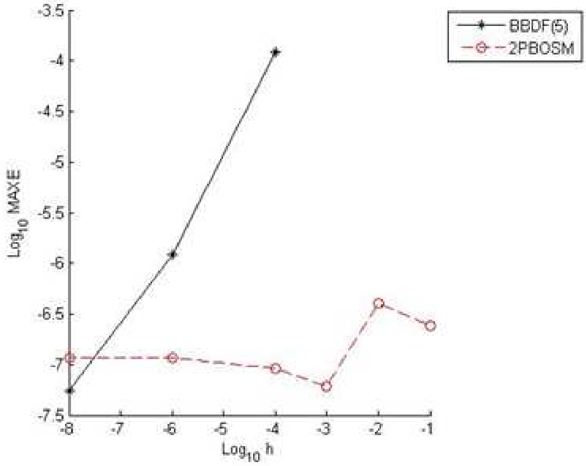


Figure 6: Graph maximum errors versus $h$ for BBDF(5) and 2PBOSM when solving problem 4

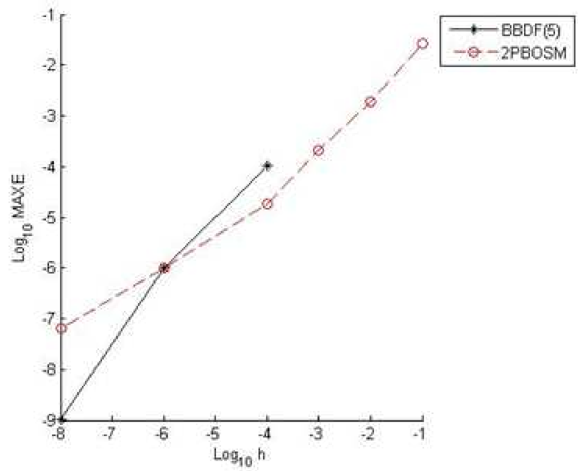


Figure 7: Graph maximum errors versus $h$ for BBDF(5) and 2PBOSM when solving problem 5.

In Table 1 - 3, 2PBOSM outperformed the 3PBEBDF by obtaining smaller maximum error. In terms of timing, the results show that the 2PBOSM is faster than 3PBEBDF in terms of execution times in microseconds. Table 4 - 5 display the comparison between the 2PBOSM and BBDF(5). The results show that the 2PBOSM outperformed the BBDF(5) in terms of maximum error at different values of step sizes. The 2PBOSM manage to solve and obtain acceptable results at larger step sizes for all the tested problems. As the step sizes reduced the accuracy for solving the tested problems improved and the solutions becoming more accurate. In Table 1 -3, we could observe that as the step size decreased the total function call increased. Figure 3 - 7 shows the plotted graph for maximum error versus different values of h compared to the existing methods.

## 6. Conclusion

For conclusion, the two-point block one-step method is suitable for solving stiff ordinary differential equation. This method also has proved that it can solve the stiff problems although at larger value of step size.

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