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EXISTENCE THEOREM FOR AN INITIAL VALUE PROBLEM OF FRACTIONAL ORDER IN $L_1(0,1)$

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Abstract: In this paper we study the existence of solution $x \in L_1(0,1)$ for a functional integral equation. As an application we prove the existence of solution for an initial value problem of the fractional order. The main tools used are Schauder fixed point theorem, Lusin theorem and Scorza Dragoni theorem.

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1. Introduction

The class of the integral equations, arises in several fields such as mathematical analysis, nonlinear functional analysis and mathematical physics [8],[9],[11],[15].

In this work, we use Schauder fixed point theorem and a set of conditions on f_1, f_2, k_1, k_2 and g to prove the existence of L_1-solution of equation

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$$x(t) = g(t) + \int_0^1 k_1(t,s) f_1\left(s, f_2\left(s, \int_0^s k_2(s,\tau)\, x(\tau)\, d\tau\right)\right) ds, \quad t \in (0,1). \quad (1)$$

The set of conditions we imposed to prove the existence of L_1-solution of equation (1), turn to be naturally satisfied in some applications. In particular, we can apply our existence result to establish the existence of L_1-solution of the initial value problem of fractional order

$$\frac{dy(t)}{dt} = g(t) + \int_0^1 k_1(t,s) f_1\left(s, f_2\left(s, D^\beta y(s)\right)\right) ds, \quad and \quad t > 0 \qquad (2)$$
$$y(0) = y_0$$

where $D^\beta y(s)$ is the fractional derivative of y and $\beta \in (0,1]$.

Sections 2,3 cover the relevant material needed in our work. The main result is given in Section 4. We discuss a special case (convolution type) of our integral equation (1) in Section 5. Finally, our application in initial value problem of fractional order is presented in Section 6.

2. Preliminaries

In this section we recall some notations and results that will be needed to establish the main result.

Define $L_p = L_p(0,1)$, $1 \le p < \infty$ as the set of all measurable functions on the interval $(0,1)$ where $|x|^p$ is Lebesgue integrable function on the interval $(0,1)$, for such function we define its norm

$$\|x\|_p = \left\{ \int_0^1 |x(t)|^p \, dt \right\}^{\frac{1}{p}}.$$

Definition 1. *Assume that $f : (0,1) \times R \to R$ satisfies Carathéodory conditions, that is $f(t,x)$ is measurable in t for any $x \in R$ and continuous in x for almost all $t \in (0,1)$, $x \in R$. Then to every function x that is measurable on the interval $(0,1)$ we may assign the function*

$$(Fx)(t) = f(t, x(t)), t \in (0,1)$$

The operator F defined in this way is called the superposition operator generated by the function f.

We remark that this operator is one of the simplest and most important operators that is investigated in nonlinear functional analysis [1],[2],[4],[5],[6],[7]. Furthermore, we have the following theorem [10],[12].

Theorem 2. *Let the function f satisfy the conditions in Definition 1. The superposition operator F generated by the function f maps continuously the space $L_1(0,1)$ into itself if and only if*

$$|f(t,x)| \leq a(t) + b|x|,$$

for all $t \in (0,1)$ and $x \in R$, where a is a function belong to $L_1(0,1)$ and b is a nonnegative constant.

Definition 3. *Suppose E is a real Banach space. A mapping $H : E \to E$ is said to be completely continuous if H is continuous and $H(Y)$ is relatively compact for every bounded subset Y of E.*

The next theorems will be used to prove our result [16],[17],[18].

Theorem 4 (Schauder)**.** *Let C be a nonempty, convex, closed, and bounded subset of a Banach space E. Let $H : C \to C$ be a completely continuous mapping . Then H has at least one fixed point in C.*

Theorem 5 (Lusin)**.** *Let $\varphi : [0,1] \to R$ be masurable function. for any $\epsilon > 0$ there is a closed subset A_ϵ of $[0,1]$, with meas.$(A_\epsilon^c) < \epsilon$, such that $\varphi|_{A_\epsilon}$ is continuous.*

Theorem 6 (Scorza Dragoni)**.** *Let $f : [0,1] \times R \to R$ be a function satisfying Caratheodory hypothesis. Then for eash $\epsilon > 0$ ther exists a closed subset D_ϵ of $[0,1]$, with meas.$([0,1] - D_\epsilon) =$ meas.$D_\epsilon^c < \epsilon$ and such that $f|_{D_\epsilon \times R}$ is continuous.*

3. Fractional Calculus

We give the definition of both differential and integral operator of fractional order. Furthermore, we give some results that are needed in the development of our main result [13],[14].

Definition 7. *Let $f \in L_1[0,1]$ and $\beta \in R_+ = [0,\infty)$. The Riemman-Liouville (R-L) fractional order integral of the function f of order β is defined as*

$$I_0^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \; 0 \leq t \leq 1.$$

Definition 8. *Let $g(t)$ be an absolutely continuous function on $[0,1]$. Then the fractional derivative of order $\alpha \in (0,1]$ of $g(t)$ is defined as*

$$D_a^{\alpha}\, g\,(t) = I_a^{1-\alpha} D\, g\,(t)\,, \quad D = \frac{d}{dt}.$$

Next, we list some basic properties of differential and integral operators of fractional order that will be used frequently in this work [5].

1. The operator I_a^{α} maps L_1 into itself continuously.
2. Let $f \in L_1$ and $\alpha, \beta \in (0,1]$, then $I_a^{\alpha} I_a^{\beta} f\,(t) = I_a^{\alpha+\beta} f\,(t)$. Furthermore, $(I_a^{\alpha})^n\, f\,(t) = I_a^{n\alpha} f\,(t)\,,\ n = 1,2,3,\cdots$.
3. Let $\alpha, \beta \in (0,1]$ and $f, Df \in L_1$, we have $DI_a^{\alpha} f\,(t) = I_a^{\alpha} Df\,(t)\,,$ when $f\,(a) = 0$.

4. Existence of Solution

In this section, we present our main result by proving the existence of solutions of the nonlinear integral equation of mixed type (1) . Let us suppose the operator,

$$Tx(t) = g(t) + \int_0^1 k_1\,(t,s)\, f_1\left(s, f_2\left(s, \int_0^s k_2\,(s,\tau)\, x\,(\tau)\, d\tau\right)\right) ds \tag{3}$$

First we impose the following conditions on the functions appearing in equation (3).

(i) $g \in L_1 = L_1(0,1)$, a.e. nonnegative on (0,1).

(ii) $f_i : (0,1) \times R \to R,\ i = 1,2$ satisfy Carathėodory condition i.e f is measurable in t for any $x \in R$ and continuous in x for almost all $t \in (0,1)$.

(iii) There are two functions $a_i \in L_1$, $i = 1,2$ and two positive constants b_i, $i = 1,2$ such that

$$|f_i\,(t, x(t))| \leq a_i\,(t) + b_i\,|x|\,, \quad i = 1,2$$

for all $t \in (0,1)$.

(*iv*) $k_i : (0,1) \times (0,1) \to R$, $i = 1,2$ are measurable and the integral operators K_i, $i = 1,2$ maps the space L_1 into itself.

(*v*) The positive constant c satisfies $c = b_1 b_2 \|K_1\| \|K_2\| < 1$.

Now we have the following lemma which is needed in proving our main result.

Lemma 9. *Let the above hypotheses* $(i) \to (v)$ *be satisfied. Then there exists a unique a.e. nonnegative function* x_0 *belonging to* L_1*, such that*

$$x_0(t) = g(t) + \int_0^1 k_1(t,s) \left\{ a_1(s) + b_1 \left(a_2(s) + b_2 \int_0^s k_2(s,\tau)\, x_0(\tau)\, d\tau \right) \right\} ds. \tag{4}$$

Proof. Set

$$\psi(t) = g(t) + \int_0^1 k_1(t,s) \{a_1(s) + b_1 a_2(s)\}\, ds$$

clearly $\psi \in L_1$ according to condition (iv). Let $B_r = \{x : x \in L_{1,} \|x\| \le r\}$ where,

$$r = \frac{\|\psi\|}{1-c} = \frac{\|\psi\|}{1 - b_1 b_2 \|K_1\| \|K_2\|}.$$

Next, we consider the operator A that is defined by

$$Ax(t) = g(t) + \int_0^1 k_1(t,s) \left\{ a_1(s) + b_1 \left(a_2(s) + b_2 \int_0^s k_2(s,\tau)\, x(\tau)\, d\tau \right) \right\} ds$$

clearly, the operator A maps the space L_1 into itself . Moreover we can get, for $x \in B_r$

$$\begin{aligned}
\|Ax\| &\le \int_0^1 \left\{ |g(t)| + \left| \int_0^1 k_1(t,s) \{a_1(s) + b_1 a_2(s)\}\, ds \right| \right\} dt + \\
&\quad + \int_0^1 \left\{ \int_0^1 b_1 b_2 |k_1(t,s)| \left(\int_0^s |k_2(s,\tau)|\, |x(\tau)|\, d\tau \right) ds \right\} dt \\
&\le \|\psi\| + b_1 b_2 \|K_1\| \|K_2\| \|x\| \\
&\le \|\psi\| + b_1 b_2 \|K_1\| \|K_2\| r
\end{aligned}$$

$$\leq \|\psi\| + b_1 b_2 \|K_1\| \|K_2\| \left(\frac{\|\psi\|}{1 - b_1 b_2 \|K_1\| \|K_2\|} \right) = r$$

This shows that A maps B_r into itself.

Next, let Q_r be subset of B_r consisting of all functions that are a.e. nonnegative on $(0,1)$, $Q_r = \{x : x \in B_r, x(t) \geq 0 \ a.e.\}$. Furthermore, Q_r is closed subset of L_1 to show this, let us take $\{x_n\}$ be a convergent sequence of elements in Q_r converging to x (in norm of $L_1(0,1)$, $\|x_n - x\|_{L_1} \to 0$) this implies that the sequence $\{x_n\}$ converges in measure to x. Using Vitali theorem [19], we deduce that the sequence$\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ that converges a.e. to x on $(0,1)$. Since $\{x_n\}$ is nonnegative sequence, then its limit x is also nonnegative on $(0,1)$. Which mean that $x \in Q_r$ and hence Q_r is closed subset of L_1 and so it is a complete metric space.

Now, we prove that $A|_{Q_r}$ is contraction mapping. Let $x_1, x_2 \in Q_r$ we have,

$$\begin{aligned}
\|Ax_1 - Ax_2\| &= \int_0^1 |Ax_1(t) - Ax_2(t)|\, dt \\
&\leq \int_0^1 \int_0^1 |k_1(t,s)| \{b_1 b_2 \int_0^1 |k_2(s,\tau)|\, d\tau \int_0^1 |x_1(\tau) - x_2(\tau)|\, d\tau\} ds dt \\
&\leq b_1 b_2 \|K_1\| \|K_2\| \|x_1 - x_2\|.
\end{aligned}$$

Since $0 < b_1 b_2 \|K_1\| \|K_2\| < 1$, then we deduce that $A|_{Q_r}$ is contraction mapping. Then there exists a unique a.e. nonnegative function x_0 belonging to L_1 that satisfied equation (4) □

Theorem 10. *Assume that the conditions* $(i) \to (v)$ *are satisfied and* k *satisfies the Caratheodory condition. Then the equation (1) has at least one solution* $x \in L_1$.

Proof. We have two cases.

First, assume that x_0 the zero element of L_1 solves equation (4) and let,

$$y(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left(s, f_2 \left(s, \int_0^s k_2(s,\tau) x_0(\tau)\, d\tau \right) \right) ds \quad t \in (0,1)$$

then from condition (iii) we get,

$$|y(t)| \leq |g(t)| + \int_0^1 |k_1(t,s)| \{ \left| a_1(s) + b_1(a_2(s) + b_2 \int_0^s k_2(s,\tau) x_0(\tau)\, d\tau) \right| \} ds.$$

$$|y(t)| \leq x_0(t), \quad a.e.$$

From Lemma 9, we get $y = 0$ and hence x_0 solve equation (1).

Second, we assume that x_0 is not the zero element of L_1 and we consider the set $Q = \{y : y \in L_{1,} \ |y(t)| \leq x_0(t), a.e\}$ This set Q can be shown to be nonempty, closed, bounded and convex in L_1.

Now let be the nonlinear operator associated with equation (1) and defined by

$$Tx(t) = g(t) + \int_0^1 k_1(t,s) f_1\left(s, f_2\left(s, \int_0^s k_2(s,\tau)x(\tau)d\tau\right)\right) ds, \quad t \in (0,1).$$

Based on our assumptions, we can see that T maps L_1 into itself continuously, we will prove that $T : Q \to Q$. Let $x \in Q$ then

$$\begin{aligned}
|Tx| &= \left| g(t) + \int_0^1 k_1(t,s) f_1\left(s, f_2\left(s, \int_0^s k_2(s,\tau)x(\tau)d\tau\right)\right) ds \right| \\
&\leq |g(t)| + \int_0^1 |k_1(t,s)| \{|a_1(s)| + b_1(|a_2(s)| + b_2 \int_0^s |k_2(s,\tau)| |x_0(\tau)| d\tau)\} ds \\
|Tx| &\leq x_0(t), \quad \text{a.e.}
\end{aligned}$$

Hence Tx belong to Q.

In the following we show that the set $T(Q)$ is relatively compact in L_1. So the Schauder fixed point principle works to get our aim. To achieve this we assume that $\{y_n\}$ is a sequence in Q and fixed $\epsilon > 0$. Upon using Theorem 5 and Theorem 6 imply that there is a closed subset D_ϵ of $[0,1]$ with means.$D_\epsilon^c < \epsilon \,(D_\epsilon^c = [0,1] - \ D_\epsilon)$ such that all the restriction $g|_{D_\epsilon}$, $x_0|_{D_\epsilon}$ and $k_1|_{D_\epsilon \times [0,1]}$, $k_2|_{D_\epsilon \times [0,1]}$ are uniformly continuous. In what follows we show that $\{y_n\}$ is equicontinuous on D_ϵ , for that let $\epsilon_1 > 0$ be given since g is uniformly continuous on D_ϵ, there exists $\delta > 0$ such that $t^{'}, t^{''} \in D_\epsilon$, $\left|t^{''} - t^{'}\right| < \delta$ implies that

$$\left|g(t^{''}) - g(t^{'})\right| < \frac{\epsilon_1}{2}$$

also $k_1|_{D_\epsilon \times [0,1]}$ is uniformly continuous, there exists $\delta > 0$ such that

$t^{'}, t^{''} \in D_\epsilon$, $\left|t^{''} - t^{'}\right| < \delta$ we have

$$\left|k_1(t^{''}, s) - k_1(t^{'}, s)\right| < \frac{\epsilon_1}{2\gamma}$$

where
$$\gamma = \|a_1\| + b_1 \|a_2\| + b_1 b_2 \|K_2\| \|x_0\| .$$
Finally, for $t^{'}, t^{''} \in D_\epsilon$, $\left|t^{''} - t^{'}\right| < \delta$ we get

$$\begin{aligned}
\left|Ty_n\left(t^{''}\right) - Ty_n\left(t^{'}\right)\right| &\leq \left|g(t^{''}) - g(t^{'})\right| + \int_0^1 \left|k_1(t^{''}, s) - k_1(t^{'}, s)\right| \cdot \\
&\quad \left|\{a_1(s) + b_1(a_2(s) + b_2 \int_0^s k_2(s,\tau)\, x_0(\tau)\, d\tau)\}\right| ds \\
&\leq \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2\gamma}\gamma = \epsilon_1.
\end{aligned}$$

Also, the sequence $\{T(y_n)\}$ is a sequence of uniformly bounded function, since .

$$\begin{aligned}
|Ty_n(t)| &\leq |g(t)| + \\
&\quad + \int_0^1 |k_1(t,s)| \{|a_1(s)| + b_1(|a_2(s)| + b_2 \int_0^s |k_2(s,\tau)|\, |x_0(\tau)|\, d\tau)\} ds \\
&\leq x_0(t), a.e.
\end{aligned}$$

Hence
$$\|T(y_n)\|_{C(D_\epsilon)} \leq \|x_0\|_{C(D_\epsilon)}, \forall n.$$
This implies that the sequence $\{T(y_n)\}$ is a sequence of uniformly bounded and equicontinuous functions on D_ϵ. Hence, in view of Ascoli-Arzěla theorem [3] we deduce that $\{T(y_n)\}$ is relatively compact subset of $C(D_\epsilon)$. Now, let us observe that the above proof does not depend on the choice of ϵ. Thus we construct a sequence $\{D_k\}$ of closed subsets of $[0,1]$ with $meas. D_k^c \to 0$ such that $\{T(y_n)\}$ is relatively compact subset of $C(D_k)$. This implies that $\{T(y_n)\}$ has a convergent subsequence in each $C(D_k)$, $k = 1, 2....$ But $C(D_k)$ is a complete metric space, hence this subsequence $\{T(y_n)\}$ is Cauchy sequence in each $C(D_k), k = 1, 2, ..,$ for any $\epsilon > 0$, and n, m arbitrary large $\|Ty_m - Ty_n\|_{C(D_k)} < \epsilon$

But we want to prove that the set $T(Q)$ is relatively compact in L_1 that is $\overline{T(Q)}$ is compact in L_1. To do this, we will prove that the sequence $\{T(y_n)\}$ is convergent sequence in L_1. Since L_1 is complete metric space, then it is sufficient to prove that the sequence $\{T(y_n)\}$ is Cauchy sequence in L_1.

Let $\eta > 0$ be given,we will show that there exists $N(\eta)$ such that

$\|Ty_m - Ty_n\|_{L_1} < \eta$, whenever $n, m > N(\eta)$. Given $\eta > 0$, there is a $\delta > 0$ such that $meas.D_\delta < \delta$ implies

$$\int\limits_{D_\delta} x_0(s)\, ds < \frac{\eta}{4}.$$

Choose $k^* \in \mathbb{N}$ with $meas.D_{k^*}^c < \delta$ and

$$\begin{aligned}
\int\limits_0^1 |(Ty_m)(t) - (Ty_n)(t)|\, dt &= \int\limits_{D_{k^*}^c} |(Ty_m)(t) - (Ty_n)(t)|\, dt + \\
&\quad + \int\limits_{D_{k^*}} |(Ty_m)(t) - (Ty_n)(t)|\, dt \\
&\leq \int\limits_{D_{k^*}} |(Ty_m)(t)| + |(Ty_n)(t)|\, dt + \\
&\quad + \|Ty_m - Ty_n\|_{C(D_{k^*}^c)} \int\limits_{D_{k^*}} dt
\end{aligned}$$

$$\begin{aligned}
\int\limits_0^1 |(Ty_m)(t) - (Ty_n)(t)|\, dt &\leq \frac{\eta}{4} + \frac{\eta}{4} + \|Ty_m - Ty_n\|_{C(D_{k^*}^c)} (1-\delta) \\
&\leq \frac{\eta}{2} + \|Ty_m - Ty_n\|_{C(D_{k^*}^c)}.
\end{aligned}$$

Choose N such that $n, m > N$ implies $\|Ty_m - Ty_n\|_{C(D_{k^*})} < \frac{\eta}{2}$. Then

$$\int\limits_0^1 |(Ty_m)(t) - (Ty_n)(t)|\, dt = \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

This means that $\{T(y_n)\}$ is a Cauchy a sequence in the space L_1 and hence the set $T(Q)$ is relatively compact in L_1. Then T has at least one fixed point. Hence there exists at least one solution $x \in L_1$ of equation (1). $\square$

5. Convolution Type Integral Equation

Assume that $k : (0,1) \to R_+$ is an measurable function. For an arbitrary function $x \in L_1$ put

$$(Kx)(t) = \int_0^t k(t-s)\,x(s)\,ds, \quad t \in (0,1).$$

This operator K is a linear integral operator of convolution type and maps L_1 into itself continuously

Now consider the following condition

(vi) $k : (0,1) \to R_+$ and $k \in L_1$.

Then we have the following corollary

Corollary 11. *Let the hypotheses $(i) \to (vi)$ be satisfied. Then there exists a unique a.e. nonnegative function x_0 belonging to L_1 such that*

$$x_0(t) = g(t) + \int_0^1 k_1(t,s)\,\{a_1(s) + b_1(a_2(s) + b_2 \int_0^s k_2(s-\tau)\,(a_2(\tau) + x_0(\tau))d\tau\}ds.$$

Furthermore, the equation

$$x(t) = g(t) + \int_0^1 k_1(t,s)\,f_1\left(s, f_2\left(s, \int_0^s k_2(s-\tau)\,x(\tau)\,d\tau\right)\right)ds, \quad t \in (0,1) \tag{5}$$

has at least one integrable solution $x \in L_1$.

6. Initial Value Problems of Fractional Order

As a special case of equation (5), we consider

$$x(t) = g(t) + \int_0^1 k_1(t,s)\,f_1\left(s, f_2\left(s, \int_0^s \frac{(\tau-s)^{-\beta}}{\Gamma(1-\beta)} x(\tau)\,d\tau\right)\right)ds, \quad t \in (0,1) \tag{6}$$

where $k_2(\tau - s) = \frac{(\tau-s)^{-\beta}}{\Gamma(1-\beta)}$. Equation (6) is an integral equation of fractional order that can be written in the form

$$x(t) = g(t) + \int_0^1 k_1(t,s)\, f_1\left(s, f_2\left(s, I^{1-\beta}x(s)\right)\right) ds. \tag{7}$$

Clearly, according to Corollary 11, equation (7) has at least one solution $x \in L_1$.

Definition 12. *By a solution of the initial value problem (2) we mean an absolutely continuous function x satisfies the initial value problem (2).*

Theorem 13. *Let $\beta \in (0,1]$. Let $k_1$and f_1 satisfy the conditions of Theorem 10. If $c = b_1 \|K_1\| < 1$, then the initial value problem (2) has at least one solution $x \in L_1$*

Proof. Let x be a solution of equation (7). Putting,

$$y(t) = \int_0^t x(\tau)\, d\tau.$$

Since x is integrable on $(0,1)$, then,

$$Dy(t) = D \int_0^t x(\tau)\, d\tau \qquad a.e.$$

where $D = \frac{d}{dt}$.

Moreover, the integral $\int_0^t x(\tau)\, d\tau$ of integrable function x is absolutely continuous then,

$$Dy(t) = DI_0^1 x(t).$$

Then we have,

$$Dy(t) = x(t) \quad a.e.$$

Furthermore, we have,

$$\begin{aligned} I^{1-\beta} Dy(t) &= I^{1-\beta}x(t) \\ D^{\beta} y(t) &= I^{1-\beta}x(t). \end{aligned}$$

Consequently, we have,

$$\frac{dy(t)}{dt} = Dy(t) = g(t) + \int_0^1 k_1\left(t,s\right) f_1\left(s, f_2\left(s, D^{\beta}y(s)\right)\right) ds.$$

Since x is integrable and absolutely continuous, then

$$\begin{aligned} Dy(\tau) &= x(\tau) \\ \int_0^t \frac{d}{dt} y\left(\tau\right) d\tau &= \int_0^t x\left(\tau\right) d\tau \\ y(t) - y_0 &= \int_0^t x\left(\tau\right) d\tau. \end{aligned}$$

Clearly, $y\left(0\right) = y_0$.

We deduced that this y is an absolutely continuous function satisfies the initial value problem (2). Hence our proof is complete. □

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