International Journal of Pure and Applied Mathematics
Volume 93 No. 4 2014, 501-509
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i4.2

HARMONIOUS AND VERTEX GRACEFUL LABELING ON PATH AND STAR RELATED GRAPHS

P. Selvaraju[1], P. Balaganesan[2 §], J. Renuka[3], M.L. Suresh[4]

[1]Department of Mathematics
Vel Tech. Multi Tech. Dr. Rangarajan Dr. Sankanthula
Engineering College
Avadi, Chennai, 600 062, INDIA

[2,4]Department of Mathematics
Hindustan University
Chennai, 603 103, INDIA

[3]Departments of Mathematics
Sri Sai Ram Engineering College
INDIA

Abstract: In this paper, we show that $B^2(n,n)$ is harmonious [4] , P_i^n is harmonious [3], $P_n \times C_m$ is vertex graceful for $n \geq 2$, and $m \geq 5, m$ is odd, $B^2(n,n)$ is vertex graceful [4] , P_i^n is vertex graceful, [3], $L_n \circ K_1$ is vertex graceful $\forall n$ and $P_n \times P_2$ is vertex graceful, n is odd [3].

AMS Subject Classification: 05C78
Key Words: harmonious graphs, harmonious labeling, vertex graceful graphs, vertex graceful labeling, bi-stars, path, Cartesian product

1. Introduction

Graph labeling, where the vertices are assigned values subject to certain conditions have often been motivated by practical problems. Labelled graphs serves as useful mathematical models for a broad range of applications such as coding

Received: October 23, 2013

§Correspondence author

theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties.They facilitate the optimal non standard encoding of integers.

All graphs in this paper are finite, simple graphs with no loops or multiple edges. The symbols V (G) and E (G) denote the vertex set and edge set of the graph G. A graph with p vertices and q edges is called G(p,q) graph. Harmonious graphs naturally arose in the study by Graham and Sloane [1] of modular version of additive base problems stemming from error correction codes. They obtained some graphs are harmonious.

Definition 1.1. A Graph G is said to be harmonious if there exist an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^{\square} : E(G) \rightarrow Z_q$ defined by $f^{\square}(uv) = (f(u) + f(v))(mod\ q)$ is a bijection from $E(G)$ onto Z_q then, f is said to be harmonious labeling of G.

Definition 1.2. A graph G with p vertices and q edges is said to be vertex graceful if a labeling $f : V(G) \rightarrow \{1, 2, 3...p\}$ exists in such a way that the induced labeling $f^{\square} : E(G) \rightarrow Z_q$ defined by $f^{\square}((u, v)) = f(u) + f(v)(mod\ q)$ is a bijection from $E(G)$ onto Z_q. The concept of vertex graceful was introduced by Lee, Pan and Tsai in 2005.

Definition 1.3. For a simple connected graph G the Square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

2. Main Results of Harmonious Labeling on Path and Star Related Graphs

Theorem 2.1. *The graph $B^2(n, n)$ is harmonious $\forall$ n.*

Proof. Consider $B^2(n, n)$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are the pendant vertices. Let G be the graph $B^2(n, n)$, then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define the vertex labeling $f : V(G) \rightarrow \{0, 1, 2, 3..., (q - 1)\}$ as follows:

$$v = 0, u = 2n + 1$$
$$v_i = i, 1 \leq i \leq n$$
$$u_i = n + i, 1 \leq i \leq n$$

Let A, B, C, D denote edge set.

$$A = \{e_i = vv_i/e_i = i : 1 \leq i \leq n\}$$

$$B =\{e_i = uv_i/e_i = (2n + i + 1)(mod\ q) : 1 \le i \le n\}$$
$$C =\{e_i = vu_i/e_i = (n + i)(mod\ q) : 1 \le i \le n\}$$
$$D =\{e_i = uu_i/e_i = (3n + i + 1)(mod\ q) : 1 \le i \le n\}$$

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the $B^2(n, n)$ is harmonious graph $\forall n$. □

Illustration 2.2. *A harmonious graph $B^2(7, 7)$ is shown in the figure 1.*

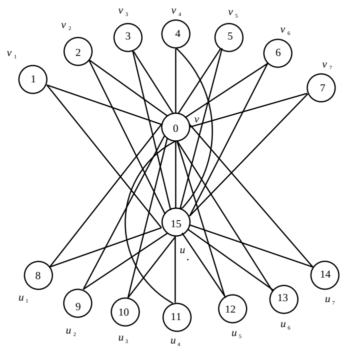

Figure 1: The graph $B^2(7, 7)$

Remark 2.3. Let α be the collection of paths P_i^n, where n is odd, and $P_i^n = u_1^i, u_2^i, ..., u_n^i, 1 \le i \le m$. Let G be the graph obtained from α with $V(G) = \cup_{i=1}^m V(p_n^i)$ and $E(G) = \cup_{i=1}^m E(p_n^i) \cup (u^i_{\frac{n+1}{2}} u^{i+1}_{\frac{n+1}{2}})$

Theorem 2.4. *The graph P_i^n is harmonious graph, $\forall n$.*

Proof. Let $G = P_i^n$ be a graph with $p = 2n$ vertices and $q = (2n - 1)$. The required vertex labeling $f : V(G) \to \{0, 1, 2, \cdots, q - 1\}$ is as follows:
case(i): n is odd

$$u_j^i = n(i - 1) + j - 1; 1 \le j \le n, i = 1, 2.$$

Let A and B denote edge set.
$A = \{e_j^1 = u_j^1 u_{j+1}^2/e_i^1 = (2n(i - 1) + 2j - 1)(mod\ q) : 1 \le j \le n - 1, i = 1, 2\}$
$B = \{e^1_{\frac{n+1}{2}} = u^1_{\frac{n+1}{2}} u^2_{\frac{n+1}{2}}/e^1_{\frac{n+1}{2}} = (2n + 1)(mod\ q)\}$
case(ii): n is even

$$u_j^i = n(i - 1) + j - 1; 1 \le j \le n, 1 \le i \le m, i \text{ is odd}$$
$$u_j^i = ni - j; 1 \le j \le n, 1 \le i \le m, i \text{ is even}$$

Let A, B, C are denote edge set.

$A = \{e_i^j = u_j^i u_j^{i+1} / e_j^i = (2n(i-1)+2j-1)(mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m$,i is odd $\}$
$B = \{e_j^i = u_j^i u_j^{i+1} / e_j^i = (2(ni-j)-1)(mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m, \quad i$ is even $\}$
$C = \{e_{\frac{n}{2}}^i = u_{\frac{n}{2}}^i u_{\frac{n}{2}}^{i+1} / e_{\frac{n}{2}}^i := (2ni-1)(mod\ q)\}$

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph G is harmonious graph $\forall n$. □

Illustration 2.5. *A harmonious graph $P_i^5 (i=1,2)$ is shown in the figure 2.*

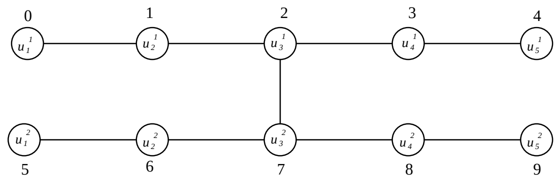

Figure 2: The graph $P_i^5 (i=1,2)$

3. Main Results of Vertex Graceful labeling on Path and Star Related Graphs

Theorem 3.1. *The graph $P_n \times C_m$ is a vertex graceful graph, $\forall n, n \geq 2$ and $m \geq 5, m$ is odd.*

Proof. Consider the graph $G = P_n \times C_m$ with nm vertices and $q = (2n-1)m$ edges. Suppose that the vertices $v_i^j; 1 \leq i \leq m$ and $j = 0, 1, 2, \cdots, n$ of the cycle C_m run consecutively with v_1^j joined to v_m^j. The required vertex labeling $f : V(G) \rightarrow 1, 2, \cdots, p$ is as follows:

$$v_i^j = \begin{cases} \dfrac{n-1}{2} + \dfrac{1+i}{2} + nj; & 1 \leq i \leq n \text{ and } i \text{ is odd}, \ 1 \leq j \leq m \\ & \text{and } j \text{ is odd} \\ \dfrac{1+i}{2} + \dfrac{nj}{2}; & 1 \leq i \leq n \text{ and } i \text{ is odd and} \\ & j = 0, 2, 4, \cdots, n \end{cases}$$

$$v_i^j = \begin{cases} \dfrac{n+1+i}{2} - \dfrac{n+1}{2} + nj; & 1 \le i \le n \; and \; i \; is \; odd, \; 1 \le j \le n \\ & and \; j \; is \; odd \\ \dfrac{n+1+i}{2} + \dfrac{nj}{2}; & 1 \le j \le n \; and \; i \; is \; odd \; and \\ & j = 0, 2, 4, \cdots, n \end{cases}$$

Let A,B,C are denote the edge set.

$A = \{e_i^j = v_i^j v_i^{j+1} = \left(\dfrac{3n+1}{2} + 2nj + i\right) (mod\ q) / 1 \le i \le n, j = 0, 1, 2, \cdots, n-1\}$

$B = \{e_i^j = v_i^j v_k^j = \left(\dfrac{n+1}{2} + 2nj + k\right) (mod\ q) / k = (1+i)(mod\ n), 1 \le i \le n,\ j = 0, 2, 4, , n\}$

$C = \{e_i^j = v_i^j v_{i+1}^j = \left(\dfrac{n+1}{2} + 2nj + i\right) (mod\ q) / 1 \le i \le n, 1 \le j \le m \; and \; j \; is \; odd\}$

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the $P_n \times C_m$ is a vertex graceful graph, for $n \ge 2$ and $m \ge 5, m$ is odd. □

Illustration 3.2. *A vertex graceful graph $P_3 \times C_5$ is shown in the figure 3.*

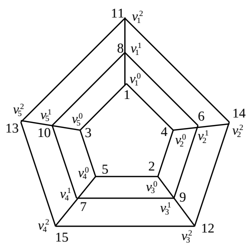

Figure 3: The graph $P_3 \times C_5$

Theorem 3.3. *The graph $B^2(n, n)$ is vertex graceful graph $\forall n$.*

Proof. Consider $B^2(n, n)$ with the vertex set $\{u, v, u_i, v_i, 1 \le i \le n\}$ where u_i, v_i are the pendant vertices. Let G be the graph $B^2(n, n)$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define the vertex labeling $f : V(G) \to \{1, 2, ..., p\}$ as follows.

case(i): n is odd

$$v = 1, u = 2n + 2$$
$$v_i = i + 1, 1 \le i \le n$$

$$u_i = n + i + 1, 1 \leq i \leq n.$$

Let A, B, C, D denote edge set.
$A = \{e_i = vv_i / e_i = i + 2 : 1 \leq i \leq n\}$
$B = \{e_i = uv_i / e_i = (2n + i + 3)(mod\ q) : 1 \leq i \leq n\}$
$C = \{e_i = vu_i / e_i = (n + i + 2)(mod\ q) : 1 \leq i \leq n\}$
$D = \{e_i = uu_i / e_i = (3n + i + 3)(mod\ q) : 1 \leq i \leq n\}$
case(ii): n is even
$u^i_j = n(i-1) + j; 1 \leq j \leq n, 1 \leq i \leq m$, i is odd
$u^i_j = ni - j + 1; 1 \leq j \leq n, 1 \leq i \leq m$, i is even
Let A, B, C are denote edge set.
$A = \{e^j_i = u^i_j u^{i+1}_j / e^i_j = (2n(i-1) + 2j + 1)(mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m$, i is odd $\}$
$B = \{e^i_j = u^i_j u^{i+1}_j / e^i_j = (2(ni - j) + 1)(mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m$, i is even $\}$
$C = \{e^i_{\frac{n}{2}} = u^i_{\frac{n}{2}} u^{i+1}_{\frac{n}{2}} / e^i_{\frac{n}{2}} := (2ni + 1)(mod\ q)\}$
It is clear that vertex set labeling and edge set labeling are distinct. Hence, the$B^2(n, n)$ is vertex graceful graph. □

Illustration 3.4. *A vertex graceful graph $B^2(7, 7)$ is shown in the figure 4.*

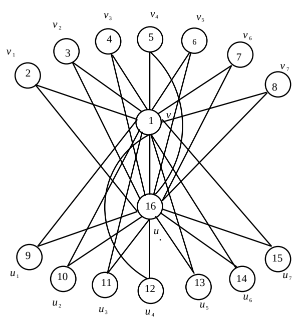

Figure 4: $B^2(7, 7)$

Theorem 3.5. *The graph P^n_i is vertex graceful graph $\forall n, n$ is odd.*

Proof. Let $G = P^i_n$ be a graph with $p = 2n$ vertices and $q = (2n - 1)$. The required vertex labeling $f : V(G) \to \{1, 2, ..., p\}$ is as follows:
case(i): n is odd

$$u_j^i = n(i-1)+j; 1 \le j \le n, i = 1,2.$$

Let A and B denote edge set.
$A = \{e_j^i = u_j^i u_{j+1}^i / e_j^i = (2n(i-1)+2j+1)(mod\ q) : 1 \le j \le n-1, i = 1,2\}$
$B = \{e^1_{\frac{n+1}{2}} = u^1_{\frac{n+1}{2}} u^2_{\frac{n+1}{2}} / e^1_{\frac{n+1}{2}} = (2n+1)(mod\ q)\}$
It is clear that vertex set labeling and edge set labeling are distinct. Hence the graph G vertex graceful $forall\ n, n$ is odd.

case(ii): n is even, $1 \le i \le m$

$$u_j^i = n(i-1)+j; 1 \le j \le n, 1 \le i \le m, \text{ i is odd}$$
$$u_j^i = ni - j + 1; 1 \le j \le n, 1 \le i \le m, \text{ i is even}$$

Let A, B, C are denote edge set.
$A = \{e_i^j = u_j^i u_j^{i+1} / e_j^i = (2n(i-1)+2j+1)(mod\ q) : 1 \le j \le n-1, 1 \le i \le m$, i is odd $\}$
$B = \left\{e_j^i = u_j^i u_j^{i+1} / e_j^i = (2(ni-j)+1)(mod\ q) : 1 \le j \le n-1, 1 \le i \le m,\right.$
i is even $\}$
$C = \{e^i_{\frac{n}{2}} = u^i_{\frac{n}{2}} u^{i+1}_{\frac{n}{2}} / e^i_{\frac{n}{2}} := (2ni+1)(mod\ q)\}$
It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph G is vertex graceful graph $\forall n$. □

Illustration 3.6. *A vertex graceful graph $P_5^i (i = 1,2)$ is shown in the figure 5.*

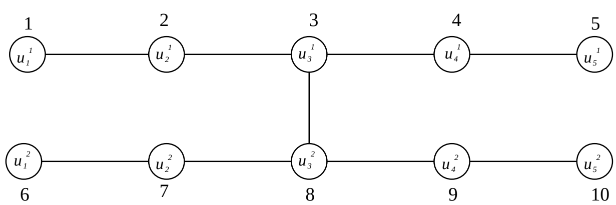

Figure 5: $P_5^i (i = 1,2)$

Definition 3.7. The graph $L_n = P_n \times P_2$ is called the ladder.

Theorem 3.8. *The graph $L_n \circ K_1$ is a vertex graceful $\forall n$.*

Proof. consider the graph $G = L_n \circ K_1$. Let $V(L_n) = u_i, v_i : 1 \le i \le n$. $E(L_n) = u_i v_i = 1 \le i \le n-1 \cup\ u_i u_{i+1} : 1 \le i \le n-1$

$\cup\, v_iv_{i+1} : 1 \leq i \leq n-1$. Let w_i be pendent vertex adjacent to u_i and let z_i be the pendent vertex adjacent to v_i. The required vertex labeling $f : V(G) \rightarrow \{1, 2, ..., p\}$ is as follows:

$$u_i = 5i - 4; 1 \leq i \leq n$$
$$v_i = 5i - 3; 1 \leq i \leq n$$
$$w_i = 5i - 5; 1 \leq i \leq n$$
$$z_i = 5i - 2; 1 \leq i \leq n$$

Let A, B, C, D, E denote the edge set.
$A = \{e_i = u_iw_i/e_i = 10i - 9(mod\ q) : 1 \leq i \leq n\}$,
$B = \{e_i = u_iv_i/e_i = 10i - 7(mod\ q), 1 \leq i \leq n\}$
$C = \{e_i = v_iz_i/e_i = 10i - 5(mod\ q) : 1 \leq i \leq n\}$,
$D = \{e_i = u_iu_{i+1}/e_i = 10i - 3(mod\ q) : 1 \leq i \leq n\}$
$E = \{e_i = v_iv_{i+1}/e_i = 10i - 1(mod\ q) : 1 \leq i \leq n\}$
It is clear that vertex set labeling and edge set labeling are distinct. Then the graph $G = L_n \circ K_1$ is vertex graceful $\forall n$. □

Theorem 3.9. *The graph $P_n \times P_2$ is a vertex graceful $\forall n$, n is odd*

Proof. Consider the graph $G = P_n \times P_2$ with $2n$ vertices and $q = 3n - 2$ edges. Let v_{1j} and v_{2j} be the first and second row vertices of G respectively for $1 \leq j \leq n$. The required vertex labeling $f : V(G) \rightarrow \{1, 2, \cdots, p\}$ is as follows:

$$v_{1j} = \frac{j+1}{2}, 1 \leq j \leq n; \text{ j is odd}$$
$$v_{1j} = \frac{n+j+1}{2}, 1 \leq j \leq n; \text{ j is even}$$
$$v_{2j} = \frac{3n+j}{2}, 1 \leq j \leq n; \text{j is odd}$$
$$v_{2j} = \frac{2n+j}{2}, 1 \leq j \leq n; \text{ j is even}$$

Let A, B, C denote edge set.
$A = \{e_j = v_{1j}v_{1j+1}/e_j = \dfrac{n+2j+3}{2}(mod\ q) : 1 \leq j \leq n-1\}$
$B = \{e_j = v_{2j}v_{2j+1}/e_j = \dfrac{5n+2j+1}{2}(mod\ q) : 1 \leq j \leq n-1\}$
$C = \{e_j = v_{1j}v_{2j}/e_j = \dfrac{3n+1+2j}{2}(mod\ q) : 1 \leq j \leq n\}$
It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph $P_n \times P_2$ is a vertex graceful $\forall n, n$ is odd. □

Illustration 3.10. *A vertex graceful graph $P_7 \times P_2$ is shown in the figure 6.*

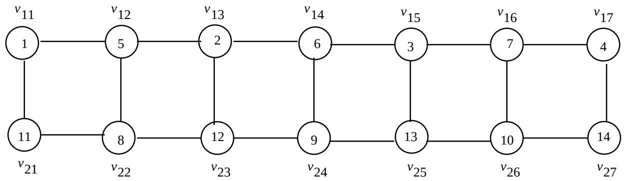

Figure 6: The graph $P_7 \times P_2$

References

[1] J.A. Gallian, A Dynamic Survey of Graph Labeling, Electronic. J. Combinatorics.

[2] Harary, F. 1972 Graph Theory Addison Wesley, Mass Reading.

[3] Ponraj. R, Studies in labeling and graphs, Ph. D., Thesis of Manonmaniam Sundaranar University, India ,September 2004.

[4] S.K Vaidya , N H shah, Graceful and odd graceful labeling of some graphs,International journal of soft computing,Vol.3,No. 1(2013) 61-68.

[5] S.K Vaidya , N H shah, Some new odd harmonious graphs, international journal of Mathematics and soft computing, Vol. I, No.1 (2011) , 9-16.