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SELF AND STRONGLY FUNCTION CHAINABLE SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper concept of self and strongly function chainable sets is introduced for topological spaces. Results proved in [8] are extended to self and strongly function chainable sets. Two characterizations of strongly function chainable sets have been established in the paper. It is shown that the space is connected if and only if it is function chainable provided the function is one to one.

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Throughout this paper X stand for topological space with topology τ and $f : X \to [0, \infty)$ will be real valued non-constant continuous function unless stated otherwise.For basic definitions refer to[5]. However to make the paper self contained definitions of function chainability between points and sets are presented below.

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Definition 1. Let (X, τ) be a topological space and if for x, y of X and $\varepsilon > 0$ there exist a non-constant continuous function $f : X \to [0, \infty)$ such that there is a sequence of $x = x_0, x_1, \ \ldots, x_n = y$ of elements of X with $|f(x_i) - f(x_{i-1})| < \varepsilon$ then this sequence is said to be an function$-f - \varepsilon-$*chain* between x and y. If x and y are function$-f \ - \ \varepsilon -$ *chainable* for every $\varepsilon > 0$ then x and y are function$-f \ - \ $*chainable.*

Definition 2. Let (X, τ) be a topological space and if for $\varepsilon > 0$ there exists a non-constant continuous function $f : X \to [0, \infty)$ such that there is function$-f \ - \ \varepsilon-$*chain* between every pair of elements x and y of X then X is said to be function$- \ f \ - \ \varepsilon -$ *chainable.* If X is function$- \ f \ - \ \varepsilon -$ *chainable* for every $\varepsilon > 0$ then X is said to be function$- \ f \ -$ *chainable.*

Definition 3. Let $A, B \subset X$.If for $\varepsilon > 0$ there exist a non-constant continuous function $f : X \to [0, \infty)$ such that there is a finite sequence $A = A_0, A_1, A_2 \ \ldots, A_n = B$ of subsets of X with $A_{i-1} \subset V_{f\varepsilon}(A_i)$ and $A_i \subset V_{f\varepsilon}(A_{i-1})$ then A and B are said to be function $- \ f \ - \ \varepsilon -$ *chainable* and this fact is denoted by $\langle A, B \rangle$. If $\langle A, B \rangle$ is $\ f \ - \ \varepsilon -$ *chainable* for every $\varepsilon > 0$ then $\langle A, B \rangle$ is said to be function$-f -$ *chainable.*

Obviously each $V_{f\varepsilon}(x)$ is an open set.

Theorem 1. *Let (X, τ) be a function$-f -$ chainable compact topological space where $f : X \to [0, \infty)$ a non-constant, continuous one to one function then X is connected.*

Proof. Suppose X is disconnected then $X = A \cup B$ where A and B are both open and closed disjoint subsets of X. Then $f(X) = f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) = \phi$ as f is one to one Since X is compact A and B are compact and hence $f(A), f(B)$ are compact subsets of $f(X)$. Also $f(A), f(B)$ are closed in $f(X)$ hence

$$|f(A) - f(B)| \ = \ \varepsilon > 0$$

As X is function$- \ f \ -$ *chainable*, $f(X)$ is chainable through points of $f(X)$. But no point of $f(A)$ and $f(B)$ can be joined by an $\varepsilon/2 -$ *chain* in $f(X)$.This is a contradiction. Or X is connected. □

Corollary 2. *Let A, B be two non-disjoint compact function$-f-$chainable subsets of X and let f be one to one function then $A \cup B$ is connected.*

The following theorem below contribute to extension of results established in [5].

Theorem 3. *Let $A, B \subset X$ and $\langle A, B\rangle$ be function$-f-$ chainable then $\langle \overline{A}, \overline{B}\rangle$ is function$-f-$ chainable.*

Proof. Let $\langle A, B\rangle$ be function$-f-$ *chainable* and $\varepsilon > 0$ Choose $\varepsilon' > 0$ such that $2\varepsilon' < \varepsilon$. Now $\langle A, B\rangle$ is function$-\ f\ -\varepsilon' -$ *chainable* or there exists a sequence $A = A_0, A_1, A_2\ \dots, A_n = B$ of subsets of X such that $A_{i-1} \subset V_{f\varepsilon'}(A_i)$ and $A_i \subset V_{f\varepsilon'}(A_{i-1})$.

Or

$$\overline{A_{i-1}} \subset \overline{V_{\ f\varepsilon'}(A_i)} \subset V_{f2\varepsilon'}(\overline{A_i}) \subset V_{f\varepsilon}(\overline{A_i})$$

and

$$\overline{A_i} \subset \overline{V_{\ f2\varepsilon'}(A_{i-1})} \subset V_{f2\varepsilon'}(\overline{A_{i-1}}) \subset V_{f\varepsilon}(\overline{A_{i-1}}).$$

Or $\langle A, B\rangle$ is function$-f-$ *chainable.* □

Theorem 4. *X is function$-f-\varepsilon-$ chainable if and only if $\langle A, B\rangle$ is function$-f-\varepsilon-$ chainable for every pair of subsets A, B of X.*

Proof. Obvious. □

Theorem 5. *Let $A \subset X$, then $\langle A, \overline{A}\rangle$ is function$-f-$ chainable.*

Proof. $A \subset \overline{A} \subset\ V_{f\varepsilon}(\overline{A})\ \forall\varepsilon > 0.$
and

$$\overline{A} = \underset{\varepsilon>0}{\cap} V_{f\varepsilon}(A)\ \forall\ \varepsilon > 0.$$

Or $\langle A, \overline{A}\rangle$ is function$-f-$ *chainable.* □

Theorem 6. *Let $A, B \subset X$. If $\langle A, C\rangle$ and $\langle B, C\rangle$ are function$-f-$ chainable then $\langle A, B\rangle$ is function$-f-$ chainable.*

Proof. Obvious. □

Next some notations are introduced.

Let $x \in X$ and $A \subset X$. Set

$$[x]_{f\varepsilon} = \{y \in X/\ y \text{ is function}-f-\varepsilon-\text{chainable to } x\},$$

and

$$[A]_{f\varepsilon} = \{B \subset X/\ \langle A, B\rangle\ \text{ is function}-f-\varepsilon-\textit{chainable}\}.$$

Now it is clear that the relation of function$-f-\varepsilon-$ *chainablility* between two points or between two subsets of X is an equivalence relation on X and partitions X into disjoint equivalence classes represented by $[x]_{f\varepsilon}$ or $[A]_{f\varepsilon}$ The set $[x]_{f\varepsilon}$ is both open and closed.

The proofs of the following theorems are obvious consequences of definitions and hence omitted.

Theorem 7. *Let $A \subset X$ and $x \in X$ If $\langle A, [x]_{f\varepsilon}\rangle$ are function$-f-\varepsilon-$ chainable then $A \subset [x]_{f\varepsilon}$.*

Note. Also $[x]_{f\varepsilon}$ is the maximal set function$-f-\varepsilon-$*chainable* to the set A and
$\sup[A]_{f\varepsilon} = [x]_{f\varepsilon}$.

Theorem 8. *If $A \subset X$, $x, y \in X$ and $\langle A, [x]_{f\varepsilon}\rangle$ and $\langle A, [y]_{f\varepsilon}\rangle$ are function$-f-\varepsilon-$ chainable then $[x]_{f\varepsilon} = [y]_{f\varepsilon}$ that is every subset A of X is function$-f-\varepsilon-$ chainable to one and only one equivalence class. Also $\langle [x]_{f\varepsilon}, [y]_{f\varepsilon}\rangle$ are function$-f-\varepsilon-$ chainable if and only if $[x]_{f\varepsilon} = [y]_{f\varepsilon}$.*

Definition 4. A map $g : X \to X$ is said to be function$-f-$*contraction* if there exists a real number $\alpha, 0 < \alpha < 1$ such that for every $x, y \in X$,
$|f(g(x)) - f(g(y))| \leq \alpha\,|f(x) - f(y)|$.

Theorem 9. *Let $g : X \to X$ and $h : X \to X$ be function$-f-$contractions then $g \circ h$ and $h \circ g$ are function$-f-$contractions.*

Theorem 10. *If $g : X \to X$ is one to one and onto and g and g^{-1} are function$-f-$contraction mappings then $[g(x)]_{f\varepsilon} = g([x]_{f\varepsilon})$.*

Proof. Let $y \in [g(x)]_{f\varepsilon}$. Then $y = y_0, y_1, y_2, \ldots, y_n = g(x)$ is function$-f-\varepsilon-$*chain* for $y_0, y_1, y_2, \ldots, y_n \in X$.

$$|f(y_i) - f(y_{i-1})| < \varepsilon, \quad 1 \leq i \leq n.$$

Since $g^{-1} : X \to X$ is function$-f-$*contraction*, for some real number $\alpha, 0 < \alpha < 1$.
$\left|f(g^{-1}(y_i)) - f(g^{-1}(y_{i-1}))\right| \leq \alpha\,|f(y_i) - f(y_{i-1})| < |f(y_i) - f(y_{i-1})| < \alpha; 1 \leq i \leq n$.
Or $g^{-1}(y) = g^{-1}(y_0), g^{-1}(y_1), \ldots, g^{-1}(y_{n-1}), x$ is function$-f-\varepsilon-$*chainable*.
Or $g^{-1}(y) \in [x]_{f\varepsilon}$. Or $y \in g([x]_{f\varepsilon})$ hence $[g(x)]_{f\varepsilon} \subset g([x]_{f\varepsilon})$.
Next $y \in g([x]_{f\varepsilon})$ then $y = g(z)$ where z is function$-f-\varepsilon-$*chainable* to x.
Or there exists $z = z_0, z_1, z_2, \ldots, z_{n-1}, z_n = x \in X$ such that $|f(z_i) - f(z_{i-1})| < \varepsilon; 1 \leq i \leq n$.
As g is function$-f-$*contraction*.

$$|f(g(z_i)) - f(g(z_{i-1}))| \leq |f(z_i) - f(z_{i-1})| < \varepsilon,$$

or $y = g(z)$ is function$-f-\varepsilon-$*chainable* to $g(x)$. Or $y \in [g(x)]_{f\varepsilon}$.
Hence $g([x]_{f\varepsilon}) \subset [g(x)]_{f\varepsilon}$ Or $[g(x)]_{f\varepsilon} = g([x]_{f\varepsilon})$. □

Next self function$-f-\varepsilon-$ *chainable* and strongly self function$-f-\varepsilon-$ *chainable* sets are defined.

Definition 5. Let $A \subset X$. If for $\varepsilon > 0$, there exist a non-constant continuous function $f : X \to [0,\infty)$ such that every two points of A can be joined by a function$-f-\varepsilon-$ *chain*, then A is said to be self function$-f-$ *chainable* if A is self function$-f-\varepsilon-$ *chainable* for every $\varepsilon > 0$.

Note.

1. The set $[x]_{f\varepsilon}$ for any $x \in X$ is always self function$-f-\varepsilon-$ *chainable*.
2. A space is function$-f-$ *chainable* if and only if it is self function$-f-$ *chainable*.

Theorem 11. *A space is self function$-f-$ chainable if and only if each of its subset is self function$-f-$ chainable.*

Theorem 12. *A space is function$-f-$ chainable if and only if each of its subset is self function$-f-$ chainable.*

Definition 6. *Let $A \subset X$. If for $\varepsilon > 0$, there exist a non-constant continuous function $f : X \to [0,\infty)$ such that every two points of A can be joined by an function$-f-\varepsilon-$chain consisting of points of A only then A is said to be strongly self function$-f-\varepsilon-$ chainable. A is strongly self function$-f-$ chainable if A is strongly self $function - f - \varepsilon -$ chainable for every $\varepsilon > 0$.*

It is simple to show that for every $\varepsilon > 0$ the concepts of funtion$-f-\varepsilon-$ *chainability* and strongly self function$-f-\varepsilon-$ *chainability* are equivalent.

Theorem 13. *Every connected set is strongly self function$-f-$ chainable if f is non-constant function on the set.*

Proof. Let $A \subset X$ and let for $\varepsilon > 0, f : X \to [0,\infty)$ be a non-constant continuous function such that it is also non-constant on A. Then $f|A : A \to [0,\infty)$ is continuous and non-constant function Now $f|A(A)$ is connected subset of $[0,\infty)$ and hence an interval which is strongly self function$-\varepsilon-$ *chainable* for every $\varepsilon > 0$. Hence for any two points $x, y \in A$ there exist points $x = x_0, x_1, x_2 \ \ldots, x_n = y \ \ in \ A$ such that $|f(x_i) - f(x_{i-1})| < \varepsilon$
Or A is strongly self function$-f-\varepsilon-$ *chainable* set. Since ε is arbitrary it follows that A is strongly self function$-f-$ *chainable*. □

Corollary 14. *Every connected space is function$-f-$ chainable.*

Proof. Obvious. □

Theorem 15. *Let (X, τ) be a topological space and $A \subset X$. If A is self function$-f-\varepsilon-$chainable then $\overline{A}$ is self function$-f-\varepsilon-$chainable.*

Proof. Refer to Theorem 5 in [8]. □

Remark. It is obvious that self function$-f-\varepsilon-$*chainability* of A is followed by self function$-f-\varepsilon-$*chainability* of $\overline{A}$.

Theorem 16. *Let $A, B \subset X$ Then:*

1. *If $(A \cap B) \neq \emptyset$ and A, B be self function$-f-$chainable then $(A \cup B)$ is self function$-f-$chainable.*
2. *If $\{A_\alpha\}_{\alpha\in\wedge}$ are self function$-f-$chainable subsets of X and*
$$\bigcap_{\alpha\in\wedge} A_\alpha \neq \emptyset$$
then
$$\bigcup_{\alpha\in\wedge} A_\alpha$$
is self function$-f-$chainable.
3. *Let E be self function$-f-$chainable and $\{E_z\}_{z\in\wedge}$ be a system of self function$-f-$chainable sets such that $(E \cap E_z) \neq \emptyset$, $\forall z \in \wedge$ and*
$$E \bigcup \left(\bigcup_{z\in\wedge} E_z \right)$$
is self$-$function$-f-$chainable.
4. *Let A, B be self function$-f-$chainable sets in X and $(A \cap B) \neq \phi$ then $\langle A, B\rangle$ is function$-f-$chainable.*

Proof. Follows from definition. □

Theorem 17. *If X is compact topological space then*
$$X = \bigcup_{i=1}^{n} [x]_{f\varepsilon}$$
. Any compact space is the finite union of disjoint collection of self function$-f-$chainable sets.

Strongly function$-\varepsilon-$*chainability* between two sets.

Definition 7. *For $\varepsilon > 0$ let $f : X \to [0, \infty)$ be a non-constant continuous function and let $A, B \subset X$. Then $\langle A, B \rangle$ is said to be strongly function$-f-\varepsilon-$chainable if and only if A and B are self function$-f-\varepsilon-$chainable and $\langle A, B \rangle$ is function$-f-\varepsilon-$chainable.*
$\langle A, B \rangle$ is said to be strongly function$-f-$chainable if it is strongly function$-f-\varepsilon-$chainable for every $\varepsilon > 0$.

Next two theorems are characterizations of strongly function$-\varepsilon-$*chainable* sets.

Theorem 18. *For $A, B \subset X$, $\langle A, B \rangle$ is strongly function$-f-\varepsilon-$chainable if and only if there exists an function$-f-\varepsilon-$chain between every point of A and every point of B.*

Proof. Let $\langle A, B \rangle$ be strongly function$-f-\varepsilon-$*chainable* and let $x \in A$ and $y \in B$. Then since $\langle A, B \rangle$ is function$-f-\varepsilon-$*chainable*, x is function$-f-\varepsilon-$*chainable* to some point z of B. By self function$-f-\varepsilon-$*chainability* of B, z is function$-f-\varepsilon-$*chainable* to y. Hence x and y are function$-f-\varepsilon-$*chainable*. Conversely let there exist an function$-f-\varepsilon-$*chain* between every point of A and every point of B. Then by theorem 2.3[5], $\langle A, B \rangle$ is function$-f-\varepsilon-$*chainable*. Next if x and x' are any two points of A then both of them are function$-f-\varepsilon-$*chainable* to every point of B and hence x and x' are function$-f-\varepsilon-$*chainable*. Or A is self function$-f-\varepsilon-$*chainable*. Likewise B is self function$-f-\varepsilon-$*chainable*. □

Theorem 19. *For $A, B \subset X$, $\langle A, B \rangle$ is strongly function$-\varepsilon-$chainable if and only if $A \cup B$ is self function$-f-\varepsilon-$chainable.*

Proof. Let $\langle A, B \rangle$ be strongly function$-f-\varepsilon-$*chainable* and let $x, y \in A \cup B$. If $x \in A$ and $y \in B$ then by theorem 18 there is an function-$f-\varepsilon-$*chain* between x and y. If $x, y \in A$ or $x, y \in B$ then as A and B are self function$-f-\varepsilon-$*chainable* sets there is an function$-f-\varepsilon-$*chain* between x and y.
Conversely suppose $A \cup B$ is self function$-f-\varepsilon-$*chainable*, let $x \in A$ and $y \in B$ then $x, y \in A \cup B$ and hence there is an function$-f-\varepsilon-$*chain* between x and y, Or $\langle A, B \rangle$ is strongly function$-f-\varepsilon-$*chainable*. □

Theorem 20. *Let $A \subset X$ and $x \in X$. If $A \subset [x]_{f\varepsilon}$ then $\left\langle A, [x]_{f\varepsilon} \right\rangle$ is strongly function$-f-\varepsilon-$chainable.*

Proof. Let $y \in A$ and $z \in [x]_{f\varepsilon}$. Then $y \in [x]_{f\varepsilon}$ and hence y and z are function$-f-\varepsilon-$*chainable*. Or $\left\langle A, [x]_{f\varepsilon} \right\rangle$ is strongly function$-f-\varepsilon-$*chainable*. □

Note. The above result shows that converse of theorem 7 holds.

Theorem 21. *Let A be self function$-f-\varepsilon-$chainable subset of X. If $\left\langle [x]_{f\ \varepsilon},\ A^c \right\rangle$ is function$-f-\varepsilon-$chainable then $\left\langle A,\ [x]^c_{f\varepsilon} \right\rangle$ is function$-f-\varepsilon-$chainable.*

Proof. Let $\left\langle [x]_{f\varepsilon},\ A^c \right\rangle$ be function$-f-\varepsilon-$*chainable* then by theorem 20 $A^c \subset [\mathrm{x}]_{\mathrm{f}\varepsilon}$ or $[x]^c_{f\varepsilon} \subset A$. Since A is self function$-f-\varepsilon-$*chainable* then $\left\langle A,\ [y]_{f\varepsilon} \right\rangle$ is function$-f-\varepsilon-$*chainable* for any $y \in A$ Or $A \subset [y]_{f\varepsilon}$ Or $[x]^c_{f\varepsilon} \subset [y]_{f\varepsilon}$. Hence by theorem 11, $\left\langle [x]^c_{f\varepsilon},\ [y]_{f\varepsilon} \right\rangle$ is function$-f-\varepsilon-$*chainable*. Or $\left\langle A,\ [x]^c_{f\varepsilon} \right\rangle$ is function$-f-\varepsilon-$*chainable*. □

Theorem 22. *A space is function$-f-\varepsilon-$chainable for every $\varepsilon > 0$ if and only if it is strongly function$-f-\varepsilon-$chainable for every $\varepsilon > 0$.*

Theorem 23. *Let A be self function$-f-\varepsilon-$chainable, then for all $y \in A$, $\left\langle A,\ [y]_{f\varepsilon} \right\rangle$ is strongly function$-f-\varepsilon-$chainable and conversely.*

Some Examples.

1. Let A and B be any two subsets of a function$-f-$*chainable* space X, then $\langle A, B\rangle$ is strongly function$-f-$*chainable*.

2. Let $(R^2,\ \mathcal{U})$ be a topological space with usual topology $\mathcal{U}$. Let $f : R^2 \to [0, \infty)$ be defined by $f(x, y) = |x|$. Then f is a non-constant continuous function on R^2.
   Further let $H_1 = \{(x,\ y) : xy = 1\}$ *and* $H_2 = \{(x,\ y) : xy = -1\}$.

   Then $H_1 \cup H_2$ is strongly function$-f-$*chainable* and consequently $\langle H_1,\ H_2\rangle$ is strongly function$-f-$*chainable*.

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