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GENERAL UNIONS OF SUNDIALS (OR LINES) IMPROVE THE HILBERT FUNCTIONS OF PROJECTIVE SCHEMES

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Abstract: A sundial $T \subset \mathbb{P}^r$ is a certain flat limit of two disjoint lines with T_{red} a reducible conic. Let $A \subset \mathbb{P}^r$, r large, the double of a linear space. We prove that a general union of A and lines or sundials has the expected postulation. Instead of A we may take other low dimensional multiple structures if certain numerical conditions on their Hilbert function are satisfied.

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1. Introduction

For any reduced closed subscheme W of a reduced scheme X and any integer $m > 0$ let (mW, X) denote the $(m-1)$-th infinitesimal neighborhood of W in X, i.e. the closed subscheme of X with $(\mathcal{I}_{W,X})^m$ as its ideal sheaf. We often write mW if $X = \mathbb{P}^r$.

For any integer $x \in \{0, \dots, r-1\}$ let $V_{x,r} \subset \mathbb{P}^r$ be an x-dimensional linear subspace. See Lemma 1 for the well-known description of the Hilbert polynomial of $mV_{x,r}$ and Lemma 4 for its Hilbert function. A sundial $A \subset \mathbb{P}^r$ is a closed subscheme such that A_{red} is a reducible conic, T, and there is a 3-

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dimensional linear linear space $V \subseteq \mathbb{P}^r$ such that $A = A \cup (2P, V)$ ([2]). A is a flat limit of a family of pairs of disjoint lines ([4]). A closed subscheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if for every integer $t > 0$ either $h^1(\mathcal{I}_X(t)) = 0$ or $h^0(\mathcal{I}_X(t)) = 0$. A general union of lines has maximal rank ([4]). A general union of sundials and lines has maximal rank ([2]). Adding a general union of many disjoint lines in $\mathbb{P}^r$, $r \ge 4$, to a zero-dimensional scheme we get a scheme with maximal rank ([1]). In a few cases (e.g, for $mV_{r,0}$, $r \ge 4$ and $2V_{0,3}$ ([3]), [1], Proposition 3, for a cohomologically similar case), this is true for an arbitrary number of general disjoint lines. In the case $r \ge 4$ it is easy to extend it to general unions of sundials and lines (see Proposition 1). We prove the following results.

Theorem 1. *Fix integers $r \ge 5$, $t \ge 0$ and $e \ge 0$. Let $X \subset \mathbb{P}^r$ be a general union of $2V_{1,r}$, t sundials and e lines. Then X has maximal rank.*

Theorem 2. *Fix integers $x \ge 2$, $r \ge 2x+4$, $t \ge 0$ and $e \ge 0$. Let $X \subset \mathbb{P}^r$ be a general union of $2V_{x,r}$, t sundials and e lines. Then X has maximal rank.*

We prove these results checking the numerical inequality coming in the following statement.

Theorem 3. *Fix an integers $r \ge 4$, $m > 0$ and a closed subscheme $Z \subset \mathbb{P}^r$ such that $\dim(Z) \le r-2$, $h^0(\mathcal{I}_Z(m-1)) = 0$ and $h^i(\mathcal{I}_Z(t)) = 0$ for all $t \ge m-i$ and all $i > 0$. Let $H \subset \mathbb{P}^r$ be a general hyperplane section. Assume $h^0(H, \mathcal{I}_{Z\cap H}(m-1)) = 0$ and $h^1(H, \mathcal{I}_{Z\cap H}(t)) = 0$ for all $i > 0$ and $t \ge m-1$. Set $f_Z(t) := h^0(\mathcal{O}_Z(t))$ and $f_{Z\cap H}(t)) := h^0(\mathcal{O}_{Z\cap H}(t))$. Assume that for all $a' \ge 0$ and $b' \ge 0$ a general union $Y \subset H$ of $Z \cap H$, a' sundials and b' lines have maximal rank. Assume*

(a) $\binom{r+k-1}{r-1}k(r-1)/r - kf_{Z\cap H}(k) + f_Z(k-1) > k^2(k+1)$ *for all* $k > m$

(b) $\binom{r+k}{r} \ge (2k+1)(k+1) + f_Z(k)$ *for all* $k \ge m$.

Then for all integers $\alpha \ge 0$ and $\beta \ge 0$ a general union $X \subset \mathbb{P}^r$ of Z, α sundials and β lines have maximal rank.

We think that the inequalities in (a) and (b) hold for $Z = mV_{x,r}$ for arbitrary m and x if $r \gg m$ and $r \gg x$. To apply Theorem 3 need weaker numerical assumptions than a corresponding statements we could prove using using only lines and a few reducible conics as in [4] instead of using sundials.

2. The schemes $mV_{x,r}$

For any line bundle $\mathcal{L}$ and any integer $a > 0$ we often write $a\mathcal{L}$ as a short-hand of $\mathcal{L}^{\oplus a}$.

Lemma 1. *Fix integers $r > x \geq 0$ and $m > 0$. Then*

$$\mathcal{O}_{mV_{x,r}} \cong \oplus_{i=0}^{m-1} \binom{r-x+i-1}{i} \mathcal{O}_{V_{x,r}}(-i)$$

as $\mathcal{O}_{V_{x,r}}$-sheaves.

Proof. We use induction on m, the case $m = 1$ being obvious. Assume $m \geq 2$ and that the lemma is true (for all x, r) for all multiplicities $m' < m$. Set $\mathcal{I} := \mathcal{I}_{V_{x,r}}$. The $\mathcal{O}_{V_{x,r}}$-sheaf $\mathcal{I}/\mathcal{I}^2$ is isomorphic to the conormal bundle of $V_{x,r}$ in $\mathbb{P}^r$. Hence $\mathcal{I}/\mathcal{I}^2 \cong (r-x)\mathcal{O}_{V_{x,r}}(-1)$. Look at the following exact sequence of $\mathcal{O}_{V_{x,r}}$ sheaves:

$$0 \to \mathcal{I}^m \to \mathcal{I}^{m-1} \to \mathcal{I}^{m-1}/\mathcal{I}^m \to 0 \tag{1}$$

Since $\mathcal{I}^{m-1}/\mathcal{I}^m \cong S^m(\mathcal{I}/\mathcal{I}^2) \cong \binom{r-x+m-1}{m}\mathcal{O}_{V_{x,r}}(-m)$ as $\mathcal{O}_{V_{x,r}}$-sheaves, (1) and the inductive assumption gives that $\mathcal{O}_{mV_{x,r}}$ is isomorphic as an $\mathcal{O}_{V_{x,r}}$-sheaf to an extension of the vector bundle $\binom{r-x+m-1}{m}\mathcal{O}_{V_{x,r}}(-m)$ by the vector bundle

$$\oplus_{i=0}^{m-2} \binom{r-x+i-1}{i} \mathcal{O}_{V_{x,r}}(-i).$$

If $x \neq 1$, then this extension splits. Now assume $x = 1$. We apply the lemma to $mV_{2,r+1}$ and then take the intersection of $mV_{2,r+1}$ with a general hyperplane of $\mathbb{P}^{r+1}$. □

Lemma 2. *We have $h^1(\mathcal{I}_{mV_{x,r}}(m)) = 0$.*

Proof. The lemma is obvious if $m = 1$. Hence we may assume $m \geq 2$ and use induction on m. The lemma is true if $x = r - 1$. Hence we may assume $r - x \geq 2$ and that the lemma is true for all integers r', x', m' with $r' - x' < r - x$. Fix a hyperplane $M \subset \mathbb{P}^r$ such that $M \supseteq V_{x,r}$. Since $\mathrm{Res}_M(mV_{x,r}) = (m-1)V_{x,r}$ and $M \cap mV_{x,r} \cong mV_{x,r-1}$, we have an exact sequence

$$0 \to \mathcal{I}_{(m-1)V_{x,r}}(m-1) \to \mathcal{I}_{mV_{x,r}}(m) \to \mathcal{I}_{mV_{x,r-1},M}(m) \to 0 \tag{2}$$

Use (2) and the inductive assumption on the integers m and $x - r$. □

Lemma 3. *Fix integers $r > x \geq 0$, $m > 0$. Then:*

(a) $h^0(\mathcal{I}_{mV_{x,r}}(t)) = 0$ *if and only if* $t \le m-1$.

(b) $h^1(\mathcal{I}_{mV_{x,r}}(t)) = 0$ *for all* $t \ge m-1$.

(c) $h^i(\mathcal{I}_{mV_{x,r}}(t)) = 0$ *for all* $i \ge 2$ *and all* $t \ge m$.

Proof. Part (a) is obvious. The case $x = 0$ of the lemma is trivial. Assume $x > 0$ and that part (b) is true for the integer $x-1$ in an $(r-1)$-dimensional projective space. Let $H \subset \mathbb{P}^r$ be a general hyperplane. We have $H \cap mV_{x,r} = (m(H \cap V_{x,r}), H)$ (as schemes). Look at the exact sequence

$$0 \to \mathcal{I}_{mV_{x,r}}(t-1) \to \mathcal{I}_{mV_{x,r}}(t) \to \mathcal{I}_{(m(V_{x,r}\cap H),H)}(t) \to 0 \tag{3}$$

Since $h^0(H, \mathcal{I}_{(m(V_{x,r}\cap H),H)}(m-1)) = 0$ and $h^1(H, \mathcal{I}_{(m(V_{x,r}\cap H),H)}(t)) = 0$ for all $t \ge m-1$ by the inductive assumption, for all $t \ge m$ the maps $H^1(\mathcal{I}_{mV_{x,r}}(t-1)) \to H^1(\mathcal{I}_{mV_{x,r}}(t))$ are surjective. Lemma 2 gives $h^1(\mathcal{I}_{mV_{x,r}}(m)) = 0$. Hence $h^1(\mathcal{I}_{mV_{x,r}}(t)) = 0$ for all $t \ge m$. Since every element of $|\mathcal{I}_{mV_{x,r}}(m))|$ is a cone with vertex containing $V_{x,r}$, the restriction map

$$H^0(\mathcal{I}_{mV_{x,r}}(m)) \to H^0(\mathcal{O}_{mV_{x,r}\cap H),H)}(m))$$

is surjective. Hence $h^1(\mathcal{I}_{mV_{x,r}}(m-1)) \le h^1(\mathcal{I}_{mV_{x,r}}(m)) = 0$. For all $i \ge 2$ we have $h^i(\mathcal{I}_{mV_{x,r}}(t)) = h^{i-1}(\mathcal{O}_{mV_{x,r}}(t))$ and the latter integer is zero if either $i \ge x+2$ or $i \le x$ or $i = x+1$ and $t \ge m-x+1$ by Lemma 1. □

Remark 1. For all integers $r > x > 0$, $t > m > 0$ we have $h^0(\mathcal{I}_{mV_{x,r}}(t)) = h^0(\mathcal{I}_{mV_{x,r}}(t-1)) + h^0(\mathcal{I}_{mV_{x-1,r-1}}(t))$ and $h^0(\mathcal{O}_{mV_{x,r}}(t)) = h^0(\mathcal{O}_{mV_{x,r}}(t-1)) + h^0(\mathcal{O}_{mV_{x-1,r-1}}(t))$ and $h^0(\mathcal{I}_{mV_{x,r}}(t)) = h^0(\mathcal{I}_{mV_{x,r}}(t-1)) + h^0(\mathcal{I}_{mV_{x-1,r-1}}(t))$ (use (3)). From Lemma 1 we get

$$h^0(\mathcal{O}_{mV_{x,r}}(t)) = \sum_{i=0}^{m-1} \binom{r-x+i-1}{i} \binom{x+t-i}{x}$$

or all $t \ge m$.

Lemma 4. *Fix integers* $r > x \ge 0$ *and* $m > 0$. *We have* $h^0(\mathcal{I}_{mV_{x,r}}(t)) = 0$ *if* $t \le m-1$ *and* $h^0(\mathcal{I}_{mV_{x,r}}(t)) = \binom{r+t}{r} - \sum_{i=0}^{m-1} \binom{r-x+i-1}{i}\binom{x+t-i}{x}$ *if* $t \ge m$.

Proof. Apply Remark 1 and parts (a) and (b) of Lemma 3. □

3. The proofs

For all integers $r \geq x+3 \geq 3$ and $k \geq m > 0$ define the integers $a_{r,x,m,k}$ and $b_{r,x,m,k}$ by the relations

$$(k+1)a_{r,x,m,k} + b_{r,x,m,k} = \binom{r+k}{r} - h^0(\mathcal{O}_{mV_{x,r}}(k)),\ 0 \leq b_{r,x,m,k} \leq k \qquad (4)$$

For the values of the integers $h^0(\mathcal{O}_{mV_{x,r}}(k))$, see Remark 1. If $x > 0$ and $k > m$, then Remark 1 and (1) for the integers k and $k-1$ give the following equality

$$\begin{aligned} &a_{r,x,m,k-1} + (k+1)(a_{r,x,m,k} - a_{r,x,m,k-1}) + b_{r,x,m,k} - b_{r,x,m,k-1} \\ &= \binom{r+k-1}{r-1} - h^0(\mathcal{O}_{mV_{x-1,r-1}}(k)) \end{aligned} \qquad (5)$$

The equality (5) is true also if $x = 0$, just setting $h^0(\mathcal{O}_{mV_{-1,r-1}}(k)) = 0$ (i.e. taking $V_{-1,n} = \emptyset$ for any $n > 0$).

Lemma 5. *Fix integers $x \geq 0$, $r \geq 4+x$, $m > 0$, $t \geq 0$ and $e \geq 0$. Let $W \subset \mathbb{P}^r$ be a general union of t sundials and e lines. Set $X := V_{x,r} \cup W$. Then either $h^0(\mathcal{I}_X(m)) = 0$ or $h^1(\mathcal{I}_X(m)) = 0$.*

Proof. Let $\ell : \mathbb{P}^r \setminus V_{x,r} \to \mathbb{P}^{r-x-1}$ be the linear projection from $V_{x,r}$. Since W is general and $r \geq x+4$, we have $V_{x,r} \cap W = \emptyset$ and $\ell(W)$ is a general disjoint union of t sundials and e lines. Since $r-x-1 \geq 3$, $\ell(W)$ has maximal rank ([2]). Hence either $h^0(\mathbb{P}^{r-x-1}, \mathcal{I}_{\ell(W)}(k)) = 0$ or $h^1(\mathbb{P}^{r-x-1}, \mathcal{I}_{\ell(W)}(k)) = 0$. Since $|\mathcal{I}_{mV_{x,r}}(m)|$ is the set of all degree m cones with vertex containing $V_{x,r}$, we have $h^0(\mathcal{I}_X(m)) = h^0(\mathbb{P}^{r-x-1}, \mathcal{I}_{\ell(W)}(k))$. We also have $h^1(\mathcal{I}_X(m)) = h^1(\mathbb{P}^{r-x-1}, \mathcal{I}_{\ell(W)}(k))$ by part (b) of Lemma 3. □

Proposition 1. *Fix integers $r \geq 4$, $m > 0$, $t \geq 0$ and $e \geq 0$. Let $X \subset \mathbb{P}^r$ be a general union of $mV_{0,r}$, t sundials and e lines. Then X has maximal rank.*

Proof. By Lemma 4 we may assume $(t, e) \neq (0, 0)$. Since a general union of a prescribed number of lines and sundials and one point has maximal rank ([2]), it is sufficient to do the case $m \geq 2$. Since a sundial is a flat limit of a family of disjoint lines ([4]), it is sufficient to do the case $e = 0$ and the case $e = 1$. Let k be the minimal positive integer such that $\binom{r+m-1}{r} + (2t+e)(k+1) \leq \binom{r+k}{r}$ (this integer is called the critical value of the triple $(r, m, 2t+e)$). Since $2t+e > 0$ we have $k \geq m$. For the case $k = m$ see the case $x = 0$ of Lemma 5. Hence we may assume $k > m$ and that Proposition 1 is true (for fixed integers r and

m) for the pairs (t', e') such that $(r, m, 2t'+e')$ has critical value $< k$. By the Castelnuovo-Mumford's lemma it is sufficient to prove that $h^1(\mathcal{I}_X(k)) = 0$ and $h^0(\mathcal{I}_X(k-1)) = 0$. Fix $P \in \mathbb{P}^r$. Since $\mathrm{Aut}(\mathbb{P}^r)$ is transitivity, without losing the generality of X we may assume that $\{P\} = V_{0,r}$. Let $H \subset \mathbb{P}^r$ be a hyperplane such that $P \notin H$.

(a) In this step we prove that $h^1(\mathcal{I}_X(k)) = 0$. Increasing if necessary either t or e we may assume $\binom{r+m-1}{r} + (2t+e)(k+1) \geq \binom{r+k}{r} - k$, i.e. we may assume that $t = \lfloor a_{r,0,m,k}/2 \rfloor$ and $e = a_{r,0,m,k} - 2t$. Set $e' := a_{r,0,m,k-1} - 2\lfloor a_{r,0,m,k-1}/2 \rfloor \in \{0, 1\}$. Let $Y \subset \mathbb{P}^r$ be a general union of $\lfloor a_{r,0,m,k-1}/2 \rfloor$ sundials and e' lines. Since $a_{r,0,m,k-1} > a_{r,0,m,k-2}$ (Lemma 7), the inductive assumption gives that $h^1(\mathcal{I}_Y(k-1)) = 0$ and $h^0(\mathcal{I}_Y(k-2)) = 0$. Since $h^1(\mathcal{I}_Y(k-1)) = 0$, we have $h^0(\mathcal{I}_Y(k-1)) = b_{r,0,m,k-1}$. Let $S \subset H$ be a general subset with $\sharp(S) = b_{r,0,m,k-1}$. Since $h^0(\mathcal{I}_Y(k-2)) = 0$, $\mathrm{Res}_H(Y) = Y$ and $h^0(\mathcal{I}_Y(k-1)) = b_{r,0,m,k-1}$, we have $h^0(\mathcal{I}_{Y\cup S}(k-1)) = 0$. The case $x = 0$ and $k' = k-1$ of (5) gives $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$.

(a1) First assume $e' = e$. Let $E \subset H$ be a general union of $t - \lfloor a_{r,0,m,k-1}/2 \rfloor$ reducible conics and write $E = E_1 \sqcup E_2$ with E_2 a general union of $b_{r,0,m,k-1}$ conics. Let $G_1 \subset H$ be a general union of sundials with $(G_1)_{\mathrm{red}} = E_1$. Let $G_2 \subset \mathbb{P}^r$ be a general union of sundials with $(G_2)_{\mathrm{red}} = E_2$. Set $X' := Y \cup G_1 \cup G_2$. Since E_2 is a general union of $b_{r,0,m,k-1}$ reducible conics contained in H, we may assume that S is the support of the nilpotent sheaf of G_2. By the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_{X'}(k)) = 0$. We have $X' \cap H = (Y \cap H) \cup G_1 \cup E_2$ and $\mathrm{Res}_H(Y') = Y \cup S$. Since $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$, it is sufficient to prove that $h^0(H, \mathcal{I}_{X'\cap H}(k))) = 0$. We have $h^0(\mathcal{O}_{X'\cap H}(k)) = a_{r,0,m,k-1} + h^0(\mathcal{I}_{G_1\cup E_2}(k)) = a_{r,0,m,k-1} + (k+1)(a_{r,0,m,k} - a_{r,0,m,k-1}) - b_{r,0,m,k-1} \leq \binom{r+k-1}{r-1}$ by (5). Since $Y \cap H$ is a general subset of H with cardinality $\sharp(Y \cap H)$, it is sufficient to prove that $h^1(H, \mathcal{I}_{G_1\cup E_2}(k)) = 0$. Let $F_2 \subset H$ be a general union of sundials with $(F_2)_{\mathrm{red}} = E_2$. Since $h^1(\mathcal{I}_{\mathrm{Res}_H(Y')}(k-1)) = 0$, a Castelnuovo's sequence shows that it is sufficient to prove that $h^1(H, \mathcal{I}_{G_1\cup F_2}(k)) = 0$. $G_1 \cup F_2$ is a general union of sundials. Hence by [2] it is sufficient to check that $h^0(\mathcal{O}_{G_1\cup F_2}(k)) \leq \binom{r+k-1}{r-1}$. By (5) it is sufficient to use that $\sharp(Y \cap H) = a_{r,0,m,k-1} \geq b_{r,0,m,k-1}$. This inequality is true by Lemma 6, because $b_{r,0,m,k-1} \leq k-1$.

(a2) Now assume $e' = 0$ and $e = 1$. Instead of $E = E_1 \sqcup E_2$ we take $E = E_1 \sqcup E_2 \sqcup L$ with L a general line of H. We work in H as in step (a1).

(a3) Now assume $e' = 1$ and $e = 0$. Write $Y = Y' \sqcup R$ with $Y' = mP \sqcup W$, W general union of $(a_{r,0,m,k-1} - 1)/2$ sundials and L a general line. Take a general $E_1 \sqcup E_2 \sqcup L \subset H$ with E_1 general union of $t - (a_{r,0,m,k-1} - 1)/2 - b_{r,0,m,k-1}$ reducible conics, E_2 a general union of $b_{r,0,m,k-1}$ reducible conics and L a general line of H through the point $R \cap H$. For a general line R the point $R \cap H$ is a general point of H. Hence $E_1 \sqcup E_2 \sqcup L$ has the Hilbert function of a general union of $t - (a_{r,0,m,k-1} - 1)/2$ reducible conics and one line. Let $L' \subset H$ be a general +line of H with L as its support and with P as the support of its ideal sheaf, i.e. a general scheme $L \cup \nu \subset H$ with $\nu \subset H$ a general degree 2 scheme with $\nu_{\mathrm{red}} = \{P\}$. Let $G_1 \subset H$ be a general union of sundials with $(G_1)_{\mathrm{red}} = E_1$. Let $G_2 \subset \mathbb{P}^r$ be a general union of sundials with $(G_2)_{\mathrm{red}} = E_2$. Set $X'' := Y \cup G_1 \cup G_2 \cup L'$. Since $R \cup L'$ is a sundial of $\mathbb{P}^r$ and $\mathrm{Res}_H(X'') = Y \cup S$, it is sufficient to prove that $h^1(H, \mathcal{I}_{X'' \cap H}(k)) = 0$. Let $F_2 \subset H$ be a general union of sundials with $(F_2)_{\mathrm{red}} = E_2$. Take a general line $D \subset H$ through $R \cap H$ and set $T := L' \cup D$, i.e. take a general sundial of H with $R \cap H$ as the support of the nilpotent sheaf and L as an irreducible component of T_{red}. In this step we need $a_{r,0,m,k-1} \geq b_{r,0,m,k-1} + k$ (see Lemma 6).

(b) In this step we prove that $h^0(\mathcal{I}_X(k-1)) = 0$. For the case $k - 1 = m$ see Lemma 5. Hence we may assume $k \geq m + 2$. By the definition of the integer k we have $2t + e > a_{r,0,m,k-1}$. Since a line is contained in a sundial, it is sufficient to do the case $2t + e = a_{3,0,m,k-1} + 1$. Let $Y' \subset \mathbb{P}^r$ be a general union of $\lfloor a_{r,0,m,k-2}/2 \rfloor$ sundials and$e'' := a_{r,0,m,k-2}/2$ lines. Since $a_{r,0,m,k-1} > a_{r,0,m,k-2}$ (Lemma 7), the inductive assumption gives that $h^1(\mathcal{I}_{Y'}(k-2)) = 0$ and $h^0(\mathcal{I}_{Y'}(k-3)) = 0$ (even if $k = m + 2$, because $h^0(\mathcal{I}_{mP}(m-1)) = 0$). Since $h^1(\mathcal{I}_Y(k-1)) = 0$, we have $h^0(\mathcal{I}_Y(k-1)) = b_{r,0,m,k-1}$. Let $S' \subset H$ be a general subset with $\sharp(S) = b_{r,0,m,k-2}$. Since $h^0(\mathcal{I}_{Y'}(k-3)) = 0$, $\mathrm{Res}_H(Y') = Y'$ and $h^0(\mathcal{I}_{Y'}(k-2)) = b_{r,0,m,k-2}$, we have $h^0(\mathcal{I}_{Y' \cup S'}(k-2)) = 0$. We make the construction in step (a) with minimal modifications. We need the same numerical checks. □

Proofs of Theorems 1, 2, and 3. Look at the proof of Proposition 1. Take Z, H and $Z \cap H$ as in Theorem 3. For all integers $k \geq m$ define the integers u_k and v_k by the relations

$$f_Z(k) + (k+1)u_k + v_k = \binom{r+k}{r}, 0 \leq v_k \leq k \tag{6}$$

From (6) for the integers k and $k-1$ we get

$$f_{Z \cap H}(k) + u_{k-1} + (k+1)(u_k - u_{k-1}) + v_k - v_{k-1} = \binom{r+k-1}{r-1} \tag{7}$$

Fix $(t, e) \in \mathbb{N}^2$ and suppose you want to prove that a general union X of Z, t sundials and e lines are maximal rank. It is sufficient to do all cases with $e \in \{0, 1\}$. Let k be the minimal integer such that $u_k \geq 2t + e$. To prove that $h^1(\mathcal{I}_X(k)) = 0$ it is sufficient to do it for the case $2t + e = u_k$. This is assumed to be true if $k = m$. Hence we may assume $k > m$ and use induction on the integer k. Condition (a) of Theorem 3 implies $u_k - u_{k-1} \geq k + 1$ for all $k > m$ (see the proof of Lemma 7). A weaker form of Condition (b) gives the inequality $u_{k-1} \geq k - 1$ used at the end of step (a1) (as well silently in (a2) and (b)). Condition (b) gives the inequality $u_{k-1} \geq 2k - 1$ used in step (a3). In the set-up of Theorems 1 and 2 we have $Z = 2V_{x,r}$, $Z \cap H = 2V_{x-1,r-1}$, $m = 2$, $u_k = a_{r,x,2,k}$ and $v_k = b_{r,x,2,k}$, with $x = 1$ for Theorem 2. The case $k = m$ of Theorems 1 and 2 is true by Lemma 5. Lemmas 9 and 10 gives that $a_{r,x,2,k} - a_{r,x,2,k-1} \geq k + 1$. Lemma 6 gives the inequality $a_{r,x,2,k-1} \geq 2k - 1$ needed to carry over step (a3). □

Lemma 6. *For all integers $r \geq 4$ and $t \geq m \geq 2$ we have $a_{r,0,m,t} \geq 2t + 1$.*

Proof. Assume $a_{r,0,m,t} \leq 2t$. We get $(2t+1)t + \binom{m+r-1}{r} \geq 1 + \binom{r+t}{r}$. Set $v(r,t,m) := \binom{r+t}{r} - \binom{r+m-1}{r} - 2t^2 - t$. It is sufficient to prove that $v(r,t,m) \geq 0$. We have $v(r,m,m) = \binom{r+m-1}{r-1} - 2m^2 - m \geq 0$ with equality if and only if $(r,m) \in \{(4,2),(4,3)\}$. We have $w(r,t) := (v(r,t+1,m) - v(r,t,m) = \binom{r+t}{r-1} - 4t - 3 \geq 0$ for all $r \geq 4$ and all $t \geq 2$. □

Lemma 7. *Fix integers $r \geq 4$ and $k \geq m \geq 2$. Then $a_{r,0,m,k} - a_{r,0,m,k-1} \geq k + 1$.*

Proof. We first check the case $k = m$. Notice that $a_{r,0,m,m-1} = 0$ and $a_{r,0,m,m} = \lfloor \binom{r+m-1}{r-1}/(m+1) \rfloor$. Hence for fixed m the integer $a_{r,0,m,m}$ is an increasing function of m. We have $a_{4,0,m,m} = (m+3)(m+2)/6$ if $m \equiv 0, 1 \pmod 3$ and $a_{4,0,m,m} = (m^2 + 5m - 2)/6$ if $m \equiv 0 \pmod 3$. Hence $a_{4,0,2,2} \geq m + 1$ and $a_{4,0,m,m} \geq m$ if and only if $m \geq 6$.

From now on we assume $k > m$. Assume $a_{r,0,m,k} - a_{r,0,m,k-1} \leq 2k$. From (5) we get

$$a_{r,0,m,k-1} + (k+1)k + b_{r,0,m,k} - b_{r,0,m,k-1} \geq \binom{r+k-1}{r-1}$$

Multiplying it by k, using (5), that $k\binom{r+k-1}{r-1} - \binom{r+k-1}{r} = \binom{r+k-1}{r-1}k(r-1)/r$, that $b_{r,0,m,k} \leq k$ and that $b_{r,0,m,k-1} \geq 0$ we get

$$k^2(k+1) + k^2 \geq \binom{r+k-1}{r-1}k(r-1)/r + \binom{r+m-1}{r} \tag{8}$$

Call $e(r,k,m)$ the difference between the right hand side and the left hand side of (8) and write $f(r,k) := e(r,k,m) - \binom{r+m-1}{r}$. To prove the lemma for the triple (r, m, k) it is sufficient to prove that $e(r, k, m) > 0$. Hence it is sufficient to prove that $f(r, k) > 0$. For a fixed integer k the function $f(r, k)$ is an increasing function of r. We have $f(5, k)/k = (k+4)(k+3)(k+2)(k+1)/32 - k^2 - 2k > 0$ for all $k > 2$. We have $f(4, k)/k = (k+3)(k+2)(k+1)/8 - k^2 - 2k > 0$ for all $k \geq 5$. Now assume $k \leq 4$. We need to check the cases $(r, m, k) \in \{(4, 2, 3), (4, 2, 4), (4, 3, 4)\}$. We have $a_{4,0,2,2} = 3$, $a_{4,0,2,3} = 7$, $a_{4,0,2,4} = 13$, $a_{4,0,3,3} = 5$, and $a_{4,0,3,4} = 11$. In all cases we have $a_{r,0,m,k} \geq a_{r,0,m,k-1} + k + 1$. □

Lemma 8. *For all integers $r \geq x + 4 \geq 5$ and $t \geq m \geq 2$ we have $a_{r,x,m,t} \geq 2t + 1$.*

Proof. It is easy to check that $a_{r,x,m,k} \geq a_{r-x,0,m,k}$. Apply Lemma 6. □

Lemma 9. *Fix integers $r \geq 2x + 4 \geq 8$ and $k \geq 3$. Then $k^2(k+2) < \binom{r+k-1}{r-1}k(r-1)/r - k\binom{x+k-1}{x-1} - k(r-x)\binom{x+k-2}{x-1} + \binom{x+k-1}{x} + (r-x)\binom{x+k-2}{x}$.*

Proof. Since the right hand side of the inequality in the lemma is k times $\binom{r+k-1}{r-1}(r-1)/r + \binom{x+k-1}{x}(x-1)/x - (r-x)\binom{x+k-2}{x-1}(x-2)/(x-1)$, it is sufficient to prove the inequality

$$k(k+2) < \binom{r+k-1}{r-1}(r-1)/r + \binom{x+k-1}{x}(x-1)/x - (r-x)\binom{x+k-2}{x-1}(x-2)/(x-1)$$

This inequality is obvious if $r \geq 2x + 4$. □

Lemma 10. *For all integers $r \geq 5$ and $k \geq 3$ we have $k(k+2) < \binom{r+k-1}{r-1}k(r-1)/r + r(k-1) + 1 - rk$.*

Proof. Set $u(r, k) := \binom{r+k-1}{r-1}k(r-1)/r - r + 1 - k(k+2)$. It is sufficient to prove that $u(r, k) > 0$. We have $u(5, k) = (k+4)(k+3)(k+2)(k+1)/32 - 4 - k(k+2) > 0$ for all $k \geq 3$. Then we use that $u(r+1, k) \geq u(r, k)$ for all $k \geq 4$ and that $u(r, 3) = (r+2)(r+1)(r-1)/6 - r - 15 > 0$ for all $r \geq 5$. □

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References

[1] E. Ballico, Postulation of a union $X \subset \mathbb{P}^r$, $r \geq 4$, of a given zero-dimensional scheme and several general lines, ISRN Geometry Volume 2013, Article ID 391721, 6 pages http://dx.doi.org/10.1155/2013/391721

[2] E. Carlini, M. V. Catalisano and A. V. Geramita, 3-dimensional sundials, Cent. Eur. J. Math. 9 (2011), no. 5, 949–971.

[3] E. Carlini, M. V. Catalisano and A. V. Geramita, Reduced and non-reduced linear spaces: lines and points, arXiv: 1308.0796.

[4] R. Hartshorne and A. Hirschowitz, Droites en position générale dans $\mathbb{P}^n$, Algebraic Geometry, Proceedings, La Rábida 1981, 169–188, Lect. Notes in Math. 961, Springer, Berlin, 1982.