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THE OSCILLATION OF THE NON-LINEAR DIFFERENTIAL EQUATIONS $\ddot{x}(t) + g(x(t))(\dot{x}(t))^2 + f(t)\dot{x}(t) + r(t)h(x(t)) = 0$

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Abstract: In this paper we study oscillation properties of second order non-linear homogeneous differential equation of the form

$$\ddot{x}(t) + g(x(t))(\dot{x}(t))^2 + f(t)\dot{x}(t) + r(t)h(x(t)) = 0.$$

An example has been given to illustrate the results.

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1. Introduction

The purpose we are concerned with the oscillation of the second order non-linear differential equation of the form

$$\ddot{x}(t) + g(x(t))(\dot{x}(t))^2 + f(t)\dot{x}(t) + r(t)h(x(t)) = 0, \tag{1.1}$$

where $f(t)$ and $r(t)$ are continuous real valued function on the interval $[\alpha, \infty)$,

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without any restriction on there signs and $\alpha \geq 0$ is a fixed non-negative real number, g$(x(t))$ and $h(x(t))$ are continuously differentiable functions on R where $g(y)f(y) > 0$ and $\frac{dh(y)}{dy} > 0$ for all $y(t) \neq 0$ and the following conditions holds for $h(y)$

$$\int^{\infty} \frac{\sqrt{\frac{dh(y)}{dy}}}{h(y)}\, dy < \infty \text{ and } \int^{-\infty} \frac{\sqrt{\frac{dh(y)}{dy}}}{h(y)}\, dy < \infty \tag{1.2}$$

and

$$\min \left\{ \inf_{y>0} \frac{\left[\int_y^{\infty} \frac{\sqrt{\frac{dhz)}{dz}}}{h(z)} dz\right]^2}{\int_y^{\infty} \frac{dz}{h(z)}} ; \inf_{y<0} \frac{\left[\int_y^{-\infty} \frac{\sqrt{f\frac{dh(z)}{dz}}}{h(z)} dz\right]^2}{\int_y^{\infty} \frac{dz}{h(z)}} \right\} > 0 \tag{1.3}$$

Our attention is concentrated only to such solution $x(t)$ of the differential equation (1.1) which exists on some interval $[\beta, \infty)$, for $\beta \geq \alpha$.

The study of the oscillation of second order non-linear differential equations has been increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for the second order non-linear differential equations.

Definition 1. A solution $x\,(t)$ of the differential equation (1.1) is said to be "nontrivial "if $x(t) \neq 0$ for at least one $t \in [\alpha, \infty)$.

Definition 2. A nontrivial solution $x\,(t)$ of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[\beta, \infty)$, for $\beta > \alpha$ otherwise it said to be "non oscillatory.

Definition 3. The differential equation (1.1) is said to be oscillatory if a nontrivial solution $x\,(t)$ is oscillatory.

Many criteria have been found which involve the behavior of the integral of a combination of the coefficients of second order nonlinear differential equations. This approach has been motivated by authors (for example see [1], [2], [3],[4],[5],[6],[7],[8] , [9] and [10] and the authors therein). where the study is done by reducing the problem to the estimate of suitable first integral.

The purpose of this paper is to present new criteria of oscillation of the differential equation (1.1).

2. Main Results

We prove the following theorem

Theorem 1. *The differential equation (1.1) is oscillatory if*

$$\lim_{t\to\infty}\int_{\alpha}^{t}(h(y)g(y)+\frac{dh(y)}{dy})\ ds=\infty, \tag{2.1}$$

and

$$\lim_{t\to\infty}\int_{\alpha}^{t}(r(s)-\frac{(f(s))^2}{4(h(y)g(y)+\frac{dh(y)}{dy})})ds=\infty\ , \tag{2.2}$$

where $h(y)$ satisfies the conditions (1.2) and (1.3) .

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on the interval $[\alpha,\infty)$, , without loss of generality its solution can be supposed such that $x(t)>0$ on $[\alpha,\infty)$.

We define

$$w(t)=-\dot{x}(t)h^{-1}(x(t))$$

Then $w(t)$ is well defined and satisfies the equation

$$\dot{w}(t)=(h(x(t)g(x(t)+\frac{dh(x(t)}{dx})(w(t))^2+f(t)w(t)+r(t) \tag{2.3}$$

Rewriting equation (2.3) we have

$$\dot{w}(t)=(h(x(t)g(x(t))+\frac{dh(x(t)}{dx})(w(t)+\frac{f(t)}{2(h(x(t)g(x(t))+\frac{dh(x(t)}{dx})}$$
$$-\frac{(f(t))^2}{4(h(x(t)g(x(t))+\frac{dh(x(t)}{dx})}+r(t). \tag{2.4}$$

Integrating both sides of the above equation from α to t we get

$$w(t)=w(\alpha)+\int_{\alpha}^{t}(h(x(s)g(x(s))+\frac{dh(x(s)}{dx})(w(s)+\frac{f(s)}{2(h(x(s)g(x(s))+\frac{dh(x(s)}{dx})})^2ds$$
$$+\int_{\alpha}^{t}(r(s)-\frac{(f(s))^2}{4(h(x(-)g(x(s))+\frac{dh(x(s)}{dx})})ds. \tag{2.5}$$

using the hypothesis of the theorem there exist $\beta>\alpha$ such that

$$w(t)\geq\int_{\beta}^{t}(h(x(s)g(x(s))+\frac{dh(x(s)}{dx})(w(s)+\frac{f(s)}{2(h(x(s)g(x(s))+\frac{dh(x(s)}{dx})})^2ds.$$

Define

$$H(t) = \int_{\beta}^{t} (h(x(s)g(x(s)) + \frac{dh(x(s)}{dx})(w(s) + \frac{f(s)}{2(h(x(s)g(x(s)) + \frac{dh(x(s)}{dx})})^2 ds. \quad (2.6)$$

Thus $w(t) \geq H(t)$.

Now differentiating equation (2.6) with respect to t we get

$$\dot{H}(t) \geq (h(x(t))g(x(t)) + \frac{dh(x(t)}{dx})(H(t))^2$$

Therefore

$$(h(x(t))g(x(t)) + \frac{dh(x(t)}{dx}) \leq \frac{\dot{H}(t)}{(H(t))^2}.$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for $t > \beta$ we get

$$\int_{\beta}^{t} (h(x(s)g(x(s)) + \frac{dh(x(s)}{dx})ds \leq \frac{1}{H(\beta)} - \frac{1}{H(t)},$$

since $H(t) > 0$. We conclude that

$$\lim_{t \to \infty} \int_{\beta}^{t} (h(x(s)g(x(s)) + \frac{dh(x(s)}{dx})ds < \frac{1}{H(\beta)}.$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory.

This completes the proof. □

3. Examples

The following examples illustrate the applicability of the theorem.

Example 1. Consider the second nonlinear order differential equation

$$\ddot{x}(t) + e^{-x(t)}(\dot{x}(t))^2 + 2t\dot{x}(t) + t^2 e^{x(t)} = 0, \tag{3.1}$$

for this differential equation we have $f(t) = 2t$ and $r(t) = t^2$; $g(x) = e^{-x}$ and $h(x) = e^{x(t)}$.

It is clear that the hypothesis of the theorem satisfies as follows

$$\begin{aligned} \lim_{t\to\infty} \int_\alpha^t (h(y)g(y) + \frac{dh(y)}{dy})\ ds &= \lim_{t\to\infty} \int_\alpha^t (1 + e^{x(s)})ds \\ \lim_{t\to\infty} [t + te^x]_\alpha^t &= \infty. \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to\infty} \int_\alpha^t (r(s) - \frac{(f(s))^2}{4(h(y)g(y) + \frac{dh(y)}{dy})})ds &= \lim_{t\longrightarrow\infty} \int_\alpha^t (s^2 - \frac{4s^2}{4(1+e^x)})ds \\ \lim_{t\longrightarrow\infty} \int_\alpha^t (s^2 + s^2e^x - s^2)ds &= \lim_{t\longrightarrow\infty} \int_\alpha^t s^2e^x ds \geq \lim_{t\longrightarrow\infty} \int_\alpha^t s^2 ds \\ &= \lim_{t\longrightarrow\infty} \left[\frac{s^3}{3}\right]_\alpha^t = \infty. \end{aligned}$$

Therefore the theorem implies that the differential equation is oscillatory.

Example 2. Consider the second nonlinear order differential equation

$$\ddot{x}(t) + \cot x\ (\dot{x}(t))^2 + e^t\dot{x}(t) + e^{2t}\cot x = 0, \tag{3.2}$$

for this differential equation we have $f(t) = e^t$ and $r(t) = e^{2t}$; $g(x) = \cot x$ and $h(x) = \cot x$.

To show the applicability of the hypothesis of the theorem

$$\begin{aligned} \lim_{t\to\infty} \int_\alpha^t (h(y)g(y) + \frac{dh(y)}{dy})\ ds &= \lim_{t\to\infty} \int_\alpha^t (\cot^2 x - \csc^2 x)ds = \lim_{t\to\infty} \int_\alpha^t ds \\ &= \lim_{t\to\infty} (t - \alpha) = \infty. \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to\infty} \int_\alpha^t (r(s) - \frac{(f(s))^2}{4(h(y)g(y) + \frac{dh(y)}{dy})})ds &= \lim_{t\to\infty} \int_\alpha^t (e^{2s} - \frac{e^{2s}}{4(\cot^2 x - \csc^2 x)^2})ds \\ &= \lim_{t\to\infty} \int_\alpha^t (e^{2s} - \frac{e^{2s}}{4})ds = \frac{3}{4} \lim_{t\to\infty} \int_\alpha^t e^{2s} ds \end{aligned}$$

$$= \frac{3}{8} \lim_{t\to\infty} e^{2s}]_{\alpha}^{t} = \frac{3}{8} \lim_{t\to\infty} \left[e^{2t} - e^{2\alpha}\right] = \infty.$$

Hence the theorem is applicable.

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