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THE FUNCTORIAL RELATIONS BETWEEN ALEXANDROV FUZZY TOPOLOGIES AND UPPER APPROXIMATION OPERATORS

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Abstract: In this paper, we investigate functorial relations between Alexandrov fuzzy topologies and upper approximation operators in complete residuated lattices. We present some examples.

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1. Introduction

The relationship between rough set theory and topological spaces was investigated in sets [8]. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-7, 9,10, 13-16]. Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior operators in complete residuated lattices. Kim [5-7] investigated the properties of upper approximation operators induced by Alexandrov fuzzy topologies in complete residuated lattices in a sense as Höhle's L-fuzzy topologies and L-fuzzy interior operators.

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In this paper, we investigate functorial relations between Alexandrov fuzzy topologies and upper approximation operators in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 2.1. [1-3] A structure $(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a commutative monoid;

(L3) an adjointness property holds,i.e., for all $x, y, z \in X$,

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z.$$

A operator $^* : L \rightarrow L$ defined by $a^* = a \rightarrow \perp$ is called *strong negations* if $a^{**} = a$. For $\alpha \in L, A \in L^X$, we denote $(\alpha \rightarrow A), (\alpha \odot A), \bar{\alpha}, \top_x, \top_x^* \in L^X$ as

$$(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x),\ (\alpha \odot A)(x) = \alpha \odot A(x),\ \bar{\alpha}(x) = \alpha,$$

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise}, \end{cases} \quad \top_x^*(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, ^*, \perp, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [9,10] Let X be a set. A function $R_X : X \times X \rightarrow L$ is called a *fuzzy preorder* if it satisfies the following conditions:

(E1) $R_X(x, x) = \top$ for all $x \in X$,

(E2) $R_X(x, y) \odot R_X(y, z) \leq R_X(x, z)$, for all $x, y, z \in X$.

Example 2.3. (1) We define a function $R_L : L \times L \rightarrow L$ as $R_L(x, y) = x \rightarrow y$. Then (L, R_L) is a fuzzy preordered set.

(2) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy preordered set.

Definition 2.4. [4-7,9] An operator $\mathbf{T} : L^X \rightarrow L$ is called an *Alexander fuzzy topology* on X iff it satisfies the following conditions: for all $\alpha \in L, A, A_i \in L^X$,

(T1) $\mathbf{T}(\overline{\alpha}) = \top$,
(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,
(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,
(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

A map $f : (X, \mathbf{T}_X) \to (Y, \mathbf{T}_Y)$ is *fuzzy continuous* if $\mathbf{T}_X(f^{-1}(B)) \geq \mathbf{T}_Y(B)$ for all $B \in L^Y$.

Definition 2.5.[5-7] A map $\mathcal{H} : L^X \to L^X$ is called an *upper approximation operator* iff it satisfies the following conditions
(H1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ for all $A \in L^X$ and $\alpha \in L$.
(H2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$ for all $A_i \in L^X$.
(H3) $A \leq \mathcal{H}(A)$,
(H4) $\mathcal{H}(\mathcal{H}(A)) \leq \mathcal{H}(A)$, for all $A \in L^X$.

Lemma 2.6. [1-3] Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.
(1) If $y \leq z$, then $x \odot y \leq x \odot z$.
(2) If $y \leq z$, then $x \to y \leq x \to z$ and $z \to x \leq y \to x$.
(3) $x \to y = \top$ iff $x \leq y$.
(4) $x \to \top = \top$ and $\top \to x = x$.
(5) $x \odot y \leq x \wedge y$.
(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.
(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.
(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.
(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.
(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.
(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.
(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.
(14) $y \to z \leq x \odot y \to x \odot z$.
(15) $x \to y \odot z \geq (x \to y) \odot z$.

Theorem 2.7. [5-7] Let $\mathbf{T} : L^X \to L$ be an Alexander fuzzy topology. Define $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. Then $\mathbf{T}^*$ is an Alexander fuzzy topology.

Theorem 2.8. [6] Let $\mathbf{T}_X$ be an Alexandrov fuzzy topology on X. Define $\mathcal{H}_{T_X} : L^X \times L \to L^X$ as follows

$$\mathcal{H}_{T_X}(A, r) = \bigwedge\{B \in L^X \mid A \leq B,\ \mathbf{T}_X(B) \geq r^*\}$$

Then we have the following properties.

(1) $\mathcal{H}_{T_X}(-,r) : L^X \to L^X$ is an upper approximation operator.

(2) If $r \leq s$, then $\mathcal{H}_{T_X}(A,s) \leq \mathcal{H}_{T_X}(A,r)$ for all $A \in L^X$.

(3) There exists a fuzzy preorder $R^r_{T_X} \in L^{X\times X}$ such that

$$\mathcal{H}_{T_X}(A,r) = \bigvee_{x\in X}(A(x) \odot R^r_{T_X}(x,y)).$$

(4) If $r \leq s$, then $R^r_{T_X} \geq R^s_{T_X}$ for all $A \in L^X$.

(5) If $\mathcal{H}_{T_X}(A,r_i) = B$ for all $i \in \Gamma \neq \emptyset$, then $\mathcal{H}_{T_X}(A, \bigwedge_{i\in\Gamma} r_i) = B$.

(6) Define $\mathbf{T}_{H_{T_X}} : L^X \to L$ as

$$\mathbf{T}_{H_{T_X}}(A) = \bigvee\{r^* \in L \mid \mathcal{H}_{T_X}(A,r) = A\}$$

Then $\mathbf{T}_{H_{T_X}} = \mathbf{T}_X$ is an Alexandrov fuzzy topology on X.

(7) There exists an Alexandrov fuzzy topology $\mathbf{T}^r_{T_X}$ such that

$$\mathbf{T}^r_{T_X}(A) = e_{L^X}(\mathcal{H}_{T_X}(A,r), A).$$

(8) If $r \leq s$, then $\mathbf{T}^r_{T_X} \leq \mathbf{T}^s_{T_X}$ for all $A \in L^X$.

(9) Define $\mathbf{T}_{T_X} : L^X \to L$ as

$$\mathbf{T}_{T_X}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r_{T_X}(A) = \top\}.$$

Then $\mathbf{T}_{T_X} = \mathbf{T}_X = \mathbf{T}_{H_{T_X}}$ is an Alexandrov fuzzy topology on X.

3. The functorial relations between Alexandrov fuzzy topologies and upper approximation operators

Theorem 3.1 Let $\mathbf{T}_X$ and $\mathbf{T}_Y$ be an Alexandrov fuzzy topologies on X and Y, respectively. Let $f : X \to Y$ be a map. Then the following statements are equivalent.

(1) $f : (X, \mathbf{T}_X) \to (Y, \mathbf{T}_Y)$ is fuzzy continuous.

(2) $f : (X, \mathbf{T}^*_X) \to (Y, \mathbf{T}^*_Y)$ is fuzzy continuous.

(3) $f^{-1}(\mathcal{H}_{T_Y}(\top_{f(x)}, r)) \geq \mathcal{H}_{T_X}(\top_x, r)$ for all $x \in X$ and $r \in L$.

(4) $f^{-1}(\mathcal{H}_{T^*_Y}(\top_{f(x)}, r)) \geq \mathcal{H}_{T^*_X}(\top_x, r)$ for all $x \in X$ and $r \in L$.

(5) There exists fuzzy preorders $R^r_{T_X} \in L^{X\times X}$ and $R^r_{T_Y} \in L^{Y\times Y}$ such that, for all $x, y \in X$,

$$R^r_{T_X}(x,y) \leq R^r_{T_Y}(f(x), f(y)).$$

(6) There exists fuzzy preorders $R^r_{T^*_X} \in L^{X\times X}$ and $R^r_{T^*_Y} \in L^{Y\times Y}$ such that, for all $x, y \in X$,

$$R^r_{T^*_X}(x,y) \le R^r_{T^*_Y}(f(x), f(y)).$$

(7) $f^{-1}(\mathcal{H}_{T_Y}(B,r)) \ge \mathcal{H}_{T_X}(f^{-1}(B), r)$ for all $B \in L^Y$ and $r \in L$.
(8) $f^{-1}(\mathcal{H}_{T^*_Y}(B,r)) \ge \mathcal{H}_{T^*_X}(f^{-1}(B), r)$ for all $B \in L^Y$ and $r \in L$.
(9) $f(\mathcal{H}_{T_X}(A,r)) \le \mathcal{H}_{T_Y}(f(A), r)$ for all $A \in L^X$ and $r \in L$.
(10) $f(\mathcal{H}_{T^*_X}(A,r)) \le \mathcal{H}_{T^*_Y}(f(A), r)$ for all $A \in L^X$ and $r \in L$.
(11) $\mathbf{T}^r_{T_X}(f^{-1}(B)) \ge \mathbf{T}^r_{T_Y}(B)$ for all $B \in L^Y$ and $r \in L$.
(12) $\mathbf{T}^r_{T^*_X}(f^{-1}(B)) \ge \mathbf{T}^r_{T^*_Y}(B)$ for all $B \in L^Y$ and $r \in L$.
(13) $\mathbf{T}_{\mathcal{H}_{T_X}}(f^{-1}(B)) \ge \mathbf{T}_{\mathcal{H}_{T_Y}}(B)$ for all $B \in L^Y$.
(14) $\mathbf{T}_{\mathcal{H}_{T^*_X}}(f^{-1}(B)) \ge \mathbf{T}_{\mathcal{H}_{T^*_Y}}(B)$ for all $B \in L^Y$.
(15) $\mathbf{T}_{T_X}(f^{-1}(B)) \ge \mathbf{T}_{T_Y}(B)$ for all $B \in L^Y$.
(16) $\mathbf{T}_{T^*_X}(f^{-1}(B)) \ge \mathbf{T}_{T^*_Y}(B)$ for all $B \in L^Y$.

Proof (1) $\Leftrightarrow$ (2).

$$\mathbf{T}^*_X(f^{-1}(B)) = \mathbf{T}_X(f^{-1}(B^*)) \ge \mathbf{T}_Y(B^*) = \mathbf{T}^*_Y(B).$$

$$\mathbf{T}_X(f^{-1}(B)) = \mathbf{T}^*_X(f^{-1}(B^*)) \ge \mathbf{T}^*_Y(B^*) = \mathbf{T}_Y(B).$$

(1) $\Rightarrow$ (3). Since $\mathbf{T}_X(f^{-1}(B)) \ge \mathbf{T}_Y(B)$ for all $B \in L^Y$, we have

$$\begin{aligned}
& f^{-1}(\mathcal{H}_{T_Y}(\top_{f(x)}, r))(y) = \mathcal{H}_{T_Y}(\top_{f(x)}, r)(f(y)) \\
& = \bigwedge\{B(f(y)) \mid \top_{f(x)} \le B,\ \mathbf{T}_Y(B) \ge r^*\} \\
& \ge \bigwedge\{f^{-1}(B)(y) \mid \top_x \le f^{-1}(B),\ \mathbf{T}_X(f^{-1}(B)) \ge r^*\} \\
& = \mathcal{H}_{T_X}(\top_x, r).
\end{aligned}$$

(3) $\Leftrightarrow$ (5). By Theorem 2.8(3) and (3), there exist fuzzy preorders $R^r_{T_X} \in L^{X\times X}$ and $R^r_{T_Y} \in L^{Y\times Y}$ with $R^r_{T_X}(x,y) = \mathcal{H}_{T_X}(\top_x)(y)$ and $R^r_{T_Y}(u,v) = \mathcal{H}_{T_Y}(\top_u)(v)$ such that

$$\begin{aligned}
& f^{-1}(\mathcal{H}_{T_Y}(\top_{f(x)}, r))(y) \ge \mathcal{H}_{T_X}(\top_x, r)(y) \\
& \text{iff } \mathcal{H}_{T_Y}(\top_{f(x)}, r))(f(y)) = R^r_{T_Y}(f(x), f(y)) \\
& \ge \mathcal{H}_{T_X}(\top_x, r)(y) = R^r_{T_X}(x,y)
\end{aligned}$$

(3) $\Rightarrow$ (7). By (3), for $B = \bigvee_{z\in Y}(B(z) \odot \top_z)$, we have

$$\begin{aligned}
& f^{-1}(\mathcal{H}_{T_Y}(B,r))(y) = \mathcal{H}_{T_Y}(B,r)(f(y)) \\
& = \mathcal{H}_{T_Y}(\textstyle\bigvee_{z\in Y}(B(z) \odot \top_z), r)(f(y)) \\
& = \textstyle\bigvee_{z\in Y}(B(z) \odot \mathcal{H}_{T_Y}(\top_z, r)(f(y)) \\
& \ge \textstyle\bigvee_{x\in X}(B(f(x)) \odot \mathcal{H}_{T_Y}(\top_{f(x)}, r)(f(y)) \\
& \ge \textstyle\bigvee_{x\in X}((f^{-1}(B)(x) \odot \mathcal{H}_{T_X}(\top_x, r)(y) \\
& = \mathcal{H}_{T_X}(f^{-1}(B), r)(y).
\end{aligned}$$

(7) $\Leftrightarrow$ (9). By (7), put $B = f(A)$. Then

$$\begin{array}{l} f^{-1}(\mathcal{H}_{T_Y}(f(A),r)) \geq \mathcal{H}_{T_X}(f^{-1}(f(A)),r) \geq \mathcal{H}_{T_X}(A,r) \\ \text{iff } \mathcal{H}_{T_Y}(f(A),r) \geq f(\mathcal{H}_{T_X}(A,r)). \end{array}$$

By (9), put $A = f^{-1}(B)$. Then

$$\begin{array}{l} f(\mathcal{H}_{T_X}(f^{-1}(B),r)) \leq \mathcal{H}_{T_Y}(f(f^{-1}(B)),r) \leq \mathcal{H}_{T_Y}(B,r) \\ \text{iff } \ \mathcal{H}_{T_X}(f^{-1}(B),r) \leq f^{-1}(\mathcal{H}_{T_Y}(B,r)). \end{array}$$

(7)$\Rightarrow$ (11).

$$\begin{array}{l} \mathbf{T}^r_X(f^{-1}(B)) = e_{L^X}(\mathcal{H}_{T_X}(f^{-1}(B),r), f^{-1}(B)) \\ = \bigwedge_{x\in X}(\mathcal{H}_{T_X}(f^{-1}(B),r)(x) \to f^{-1}(B)(x)) \\ \geq \bigwedge_{x\in X}(f^{-1}(\mathcal{H}_{T_Y}(B,r))(x) \to B(f(x))) \\ \geq \bigwedge_{y\in Y}(\mathcal{H}_{T_Y}(B,r)(y) \to B(y)) \\ = \mathbf{T}^r_{T_Y}(B). \end{array}$$

(11)$\Rightarrow$ (13). By (11), since $\mathcal{H}_{T_Y}(B,r) = B$ implies $\mathcal{H}_{T_X}(f^{-1}(B),r) = f^{-1}(B)$, the result hold.

(13) $\Leftrightarrow$ (15). Since $\mathbf{T}^r_{T_X}(A) = e_{L^X}(\mathcal{H}_{T_X}(A,r),A) = \top$ iff $A = \mathcal{H}_{T_X}(A,r)$, the result holds.

(15) $\Leftrightarrow$ (1). By Theorem 2.8 (9), it easily proved from $\mathbf{T}_{T_X} = \mathbf{T}_X$ and $\mathbf{T}_{T_Y} = \mathbf{T}_Y$.

Other cases are similarly proved.

Example 3.2. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation defined by

$$x \odot y = (x + y - 1) \vee 0, \ \ x \to y = (1 - x + y) \wedge 1, \ x^* = 1 - x.$$

Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$ be a set. Define a map $f : X \to Y$ as

$$f(a) = f(b) = x, \ f(c) = y, \ f(d) = z.$$

Define fuzzy preorders $R_X \in L^{X\times X}$ and $R_Y \in L^{Y\times Y}$ as follows:

$$R_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.8 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $R_X(a,b) \le R_Y(f(a), f(b))$ for all $a, b \in X$. Define

$$\mathbf{T}_Y(B) = \bigwedge_{y\in Y}(\bigvee_{x\in X}(B(x)\odot R(x,y)) \to B(y)).$$

Then $\mathbf{T}_Y$ is an Alexandrov fuzzy topology from:

(T1) $\mathbf{T}_Y(\overline{\alpha}) = \top$,

(T2) By Lemma 2.6 (8), we have

$$\begin{aligned}
&\mathbf{T}_Y(\textstyle\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{y\in Y}(\bigvee_{x\in X}(\bigwedge_{i\in\Gamma} B_i(x)\odot R(x,y)) \to \bigwedge_{i\in\Gamma} B_i(y))\\
&\ge \textstyle\bigwedge_{i\in\Gamma}\bigwedge_{y\in Y}(\bigvee_{x\in X}(B_i(x)\odot R(x,y)) \to B_i(y)) = \bigwedge_{i\in\Gamma}\mathbf{T}_Y(B_i),\\
&\mathbf{T}_Y(\textstyle\bigvee_{i\in\Gamma} B_i) = \bigwedge_{y\in Y}(\bigvee_{x\in X}(\bigvee_{i\in\Gamma} B_i(x)\odot R(x,y)) \to \bigwedge_{i\in\Gamma} B_i(y))\\
&\ge \textstyle\bigwedge_{i\in\Gamma}\bigwedge_{y\in Y}(\bigvee_{x\in X}(B_i(x)\odot R(x,y)) \to B_i(y)) = \bigwedge_{i\in\Gamma}\mathbf{T}_Y(B_i).
\end{aligned}$$

(T3) $\mathbf{T}_Y(\alpha\odot B) = \bigwedge_{y\in Y}(\bigvee_{x\in X}(\alpha\odot B(x)\odot R(x,y)) \to \alpha\odot B(y)) \ge \bigwedge_{y\in Y}(\bigvee_{x\in X}(B(x)\odot R(x,y)) \to B(y)) = \mathbf{T}_Y(A)$ from Theorem 2.6 (14).

(T4) $\mathbf{T}_Y(\alpha\to B) = \bigwedge_{y\in Y}(\bigvee_{x\in X}((\alpha\to B)(x)\odot R(x,y)) \to (\alpha\to B)(y)) \ge \bigwedge_{y\in Y}(\bigvee_{x\in X}((\alpha\to(B(x)\odot R(x,y))) \to (\alpha\to B)(y)) \ge \bigwedge_{y\in Y}(\bigvee_{x\in X}(B(x)\odot R(x,y)) \to B(y)) = \mathbf{T}_Y(A)$ from Theorem 2.6 (10,15).

$$\begin{aligned}
&\mathbf{T}_Y(B) = \textstyle\bigwedge_{y\in Y}(\bigvee_{x\in X}(B(x)\odot R(x,y)) \to B(y))\\
&= (B(x)\to B(x))\wedge(B(y)\to B(y))\wedge((0.8\odot B(y))\vee B(z)\to B(z))\\
&= (1.2 - B(y) + B(z))\wedge 1.\\
&\mathbf{T}_X(f^{-1}(B)) = \textstyle\bigwedge_{b\in X}(\bigvee_{a\in X}(f^{-1}(B)(a)\odot R_X(a,b)) \to f^{-1}(B)(b))\\
&= (B(x)\to B(x))\wedge(B(y)\to B(y))\wedge((0.6\odot B(y))\vee B(z)\to B(z))\\
&= (1.4 - B(y) + B(z))\wedge 1.
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}^*_Y(B) = \mathbf{T}_Y(B^*) = \textstyle\bigwedge_{y\in Y}(\bigvee_{x\in X}(B^*(x)\odot R(x,y)) \to B^*(y))\\
&= (B^*(x)\to B^*(x))\wedge(B^*(y)\to B^*(y))\wedge((0.8\odot B^*(y))\vee B^*(z)\to B^*(z))\\
&= (1.2 - B^*(y) + B^*(z))\wedge 1 = (1.2 - B(z) + B(y))\wedge 1.\\
&\mathbf{T}^*_Y(B) = \textstyle\bigwedge_{y\in Y}(\bigvee_{x\in X}(B^*(x)\odot R(x,y)) \to B^*(y))\\
&= \textstyle\bigwedge_{x,y\in Y}(R(x,y)\to(B^*(x)\to B^*(y)))\\
&= \textstyle\bigwedge_{x,y\in Y}(R(x,y)\to(B(y)\to B(x)))\\
&= \textstyle\bigwedge_{x\in Y}(\bigvee_{y\in X}(B(y)\odot R(x,y)) \to B(x))\\
&\mathbf{T}^*_X(f^{-1}(B)) = \textstyle\bigwedge_{b\in X}(\bigvee_{a\in X}(f^{-1*}(B)(a)\odot R_X(a,b)) \to f^{-1*}(B)(b))\\
&= (B^*(x)\to B^*(x))\wedge(B^*(y)\to B^*(y))\wedge((0.6\odot B^*(y))\vee B^*(z)\to B^*(z))\\
&= (1.4 - B^*(y) + B^*(z))\wedge 1 = (1.4 - B(z) + B(y))\wedge 1.
\end{aligned}$$

Hence $\mathbf{T}_X(f^{-1}(B)) \ge \mathbf{T}_Y(B)$ and $\mathbf{T}^*_X(f^{-1}(B)) \ge \mathbf{T}^*_Y(B)$ for all $B\in L^Y$ and $r\in L$.

Since $\mathcal{H}_{T_Y}(1_y,r)(z) = \bigwedge\{B(z) \in L^X \mid 1_y \le B,\ \mathbf{T}_Y(B) \ge r^*\}$, then $\mathcal{H}_{T_Y^*}(1_y,r)(z) = \mathcal{H}_{T_Y}(1_z,r)(y)$ from:

$$\begin{array}{lll} \mathcal{H}_{T_Y}(1_x,r)(x)=1 & \mathcal{H}_{T_Y}(1_x,r)(y)=0 & \mathcal{H}_{T_Y}(1_x,r)(z)=0 \\ \mathcal{H}_{T_Y}(1_y,r)(x)=0 & \mathcal{H}_{T_Y}(1_y,r)(y)=1 & \mathcal{H}_{T_Y}(1_y,r)(z)=(0.8-r)\vee 0 \\ \mathcal{H}_{T_Y}(1_z,r)(x)=0 & \mathcal{H}_{T_Y}(1_z,r)(y)=0 & \mathcal{H}_{T_Y}(1_z,r)(z)=1 \end{array}$$

$$\begin{array}{lll} \mathcal{H}_{T_Y^*}(1_x,r)(x)=1 & \mathcal{H}_{T_Y^*}(1_x,r)(y)=0 & \mathcal{H}_{T_Y^*}(1_x,r)(z)=0 \\ \mathcal{H}_{T_Y^*}(1_y,r)(x)=0 & \mathcal{H}_{T_Y^*}(1_y,r)(y)=1 & \mathcal{H}_{T_Y^*}(1_y,r)(z)=0 \\ \mathcal{H}_{T_Y^*}(1_z,r)(x)=0 & \mathcal{H}_{T_Y^*}(1_z,r)(y)=(0.8-r)\vee 0 & \mathcal{H}_{T_Y^*}(1_z,r)(z)=1. \end{array}$$

Similarly, we have $\mathcal{H}_{T_X^*}(1_a,r)(b) = \mathcal{H}_{T_X}(1_b,r)(a)$ for all $a,b \in X$ as

$$(\mathcal{H}_{T_X}(1_a,r)(b)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (0.6-r)\vee 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence $f^{-1}(\mathcal{H}_{T_Y}(\top_{f(a)},r)) \ge \mathcal{H}_{T_X}(\top_a,r)$ and $f^{-1}(\mathcal{H}_{T_Y^*}(\top_{f(a)},r)) \ge \mathcal{H}_{T_X^*}(\top_a,r)$ for all $x \in X$ and $r \in L$. Since $R^r_{T_Y}(x,y) = \mathcal{H}_{T_Y}(\top_x,r)(y)$ $R^r_{T_X}(a,b) = \mathcal{H}_{T_X}(\top_a,r)(b)$, we have

$$R^r_{T_Y^*}(x,y) = R^r_{T_Y}(y,x),\ R^r_{T_X^*}(a,b) = R^r_{T_X}(b,a).$$

Moreover, for all $a,b \in X$,

$$R^r_{T_X}(a,b) \le R^r_{T_Y}(f(a),f(b)),\ \ R^r_{T_X^*}(a,b) \le R^r_{T_Y^*}(f(a),f(b)).$$

$$R^r_{T_X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (0.6-r)\vee 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R^r_{T_Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (0.8-r)\vee 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $B = \bigvee_{x\in Y}(B(x)\odot 1_x)$, we have

$$\begin{aligned} &\mathcal{H}_{T_Y}(B,r) = \mathcal{H}_{T_Y}(\textstyle\bigvee_{x\in Y}(B(x)\odot 1_x),r) \\ &= \textstyle\bigvee_{x\in Y}(B(x)\odot \mathcal{H}_{T_Y}(1_x,r)) = \bigvee_{x\in Y}(B(x)\odot R^r_{T_Y}(x,-)) \\ &= (B(x),B(y),(B(y)-0.2-r)\vee B(z)). \end{aligned}$$

$$\begin{aligned} &\mathcal{H}_{T_Y^*}(B,r)(y) = \mathcal{H}_{T_Y^*}(\textstyle\bigvee_{x\in Y}(B(x)\odot 1_x),r)(y) \\ &= \textstyle\bigvee_{x\in Y}(B(x)\odot \mathcal{H}_{T_Y^*}(1_x,r)) = \bigvee_{x\in Y}(B(x)\odot R^r_{T_Y}(-,x)) \\ &= (B(x),B(y)\vee(B(z)-0.2-r),B(z)). \end{aligned}$$

$$\mathcal{H}_{T_X}(A,r) = (A(a), A(b), A(c), (A(c) - 0.4 - r) \vee A(d)),$$
$$\mathcal{H}_{T_X^*}(A,r) = (A(a), A(b), A(c) \vee (A(d) - 0.4 - r), A(d)).$$

$$\begin{aligned}
&\mathcal{H}_{T_X}(f^{-1}(B), r)) = \bigvee_{a\in X}(f^{-1}(B)(a) \odot R^r_{T_X}(a,-)) \\
&= (B(x), B(x), B(y), (B(y) - 0.4 - r) \vee B(z)) \\
&\le f^{-1}(\mathcal{H}_{T_X}(B,r)) = f^{-1}(\bigvee_{x\in Y}(B(x) \odot R^r_{T_Y}(x,-)) \\
&= (B(x), B(x), B(y), (B(y) - 0.2 - r) \vee B(z)).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{H}_{T_X^*}(f^{-1}(B), r)) = \bigvee_{a\in X}(f^{-1}(B)(a) \odot R^r_{T_X}(-,a)) \\
&= (B(x), B(x), B(y) \vee (B(z) - 0.4 - r), B(z)) \\
&\le f^{-1}(\mathcal{H}_{T_X^*}(B,r)) = f^{-1}(\bigvee_{x\in Y}(B(x) \odot R^r_{T_Y}(-,x)) \\
&= (B(x), B(x), B(y) \vee (B(z) - 0.2 - r), B(z)).
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}^r_{T_Y}(B) = \bigwedge_{y\in Y}(\mathcal{H}_{T_Y}(B,r)(y) \to B(y)) \\
&= ((B(y) - 0.2 - r) \vee B(z)) \to B(z) = (1.2 + r - B(y) + B(z)) \wedge 1
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}^r_{T_X}(A) = \bigwedge_{a\in X}(\mathcal{H}_{T_X}(A,r)(a) \to A(a)) \\
&= ((A(c) - 0.4 - r) \vee A(d)) \to A(d) = (1.4 + r - A(c) + A(d)) \wedge 1
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}^r_{T_Y^*}(B) = \bigwedge_{y\in Y}(\mathcal{H}_{T_Y^*}(B,r)(y) \to B(y)) \\
&= (B(y) \vee (B(z) - 0.2 - r)) \to B(y) = (1.2 + r - B(z) + B(y)) \wedge 1
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}^r_{T_X^*}(A) = \bigwedge_{a\in X}(\mathcal{H}_{T_X^*}(A,r)(a) \to A(a)) \\
&= (A(c) \vee (A(d) - 0.4 - r)) \to A(c) = (1.4 + r - A(d) + A(c)) \wedge 1
\end{aligned}$$

$$\mathbf{T}^r_{T_X}(f^{-1}(B)) = (1.4 + r - B(y) + B(z)) \wedge 1$$
$$\mathbf{T}^r_{T_X^*}(f^{-1}(B)) = (1.4 + r - B(z) + B(y)) \wedge 1$$

Hence $\mathbf{T}^r_{T_X}(f^{-1}(B)) \ge \mathbf{T}^r_{T_Y}(B)$ and $\mathbf{T}^r_{T_X^*}(f^{-1}(B)) \ge \mathbf{T}^r_{T_X^*}(B)$ for all $B \in L^Y$ and $r \in L$.

$$\begin{aligned}
&\mathbf{T}_{H_{T_Y}}(B) = \bigvee\{r^* \in L \mid \mathcal{H}_{T_Y}(B,r) = B\} \\
&= \bigvee\{r^* \in L \mid (B(x), B(y), (B(y) - 0.2 - r) \vee B(z)) = (B(x), B(y), B(z))\} \\
&= \bigvee\{r^* \in L \mid B(y) - 0.2 - r \le B(z)\} = (1.2 - B(y) + B(z)) \wedge 1 = \mathbf{T}_Y(B).
\end{aligned}$$

$$\begin{aligned}
&\mathbf{T}_{H_{T_X}}(f^{-1}(B)) = \bigvee\{r^* \in L \mid \mathcal{H}_{T_Y}(f^{-1}(B), r) = f^{-1}(B)\} \\
&= \bigvee\{r^* \in L \mid B(y) - 0.4 - r \le B(z)\} = (1.4 - B(y) + B(z)) \wedge 1.
\end{aligned}$$

Hence $\mathbf{T}_{H_{T_X}}(f^{-1}(B)) \ge \mathbf{T}_{H_{T_Y}}(B)$ for all $B \in L^Y$ and $r \in L$.

$$\begin{aligned}\mathbf{T}_{T_Y}(B) &= \bigvee\{r^* \in L \mid \mathbf{T}^r_{T_Y}(B) = (1.2 + r - B(y) + B(z)) \wedge 1 = 1\}\\ &= (1.2 - B(y) + B(z)) \wedge 1 = \mathbf{T}_Y(B),\\ \mathbf{T}_{T^*_Y}(B) &= \bigvee\{r^* \in L \mid \mathbf{T}^r_{T^*_Y}(B) = (1.2 + r - B(z) + B(y)) \wedge 1 = 1\}\\ &= (1.2 - B(z) + B(y)) \wedge 1 = \mathbf{T}^*_Y(B).\end{aligned}$$

$$\begin{aligned}&\mathbf{T}_{T_X}(f^{-1}(B)) = \bigvee\{r^* \in L \mid \mathbf{T}^r_{T_X}(f^{-1}(B)) = (1.4 + r - B(y) + B(z)) \wedge 1 = 1\}\\ &= (1.4 - B(y) + B(z)) \wedge 1 = \mathbf{T}_X(f^{-1}(B)),\\ &\mathbf{T}_{T^*_X}(f^{-1}(B)) = \bigvee\{r^* \in L \mid \mathbf{T}^r_{T^*_X}(f^{-1}(B)) = (1.4 + r - B(z) + B(y)) \wedge 1 = 1\}\\ &= (1.4 - B(z) + B(y)) \wedge 1 = \mathbf{T}^*_X(f^{-1}(B)).\end{aligned}$$

Hence $\mathbf{T}_{T_X}(f^{-1}(B)) \geq \mathbf{T}_{T_Y}(B)$ and $\mathbf{T}_{T^*_X}(f^{-1}(B)) \geq \mathbf{T}_{T^*_Y}(B)$ for all $B \in L^Y$ and $r \in L$.

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