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SOLUTIONS OF FUZZY RELATION EQUATIONS IN GENERALIZED RESIDUATED LATTICES

Yong Chan Kim
Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: In this paper, we investigate solutions of various types of fuzzy relation equations $A_i \circ R = B_i$, $R \circ A_i = B_i$, $A_i \Rightarrow R = B_i$ and $A_i \rightarrow R = B_i$ in generalized residuated lattice.

We give also some examples.

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1. Introduction

Sanchez [14] introduced the theory of fuzzy relation equations with various types of composition: max-min, min-max, min-α. Fuzzy relation equations with new types of composition(pseudo t-norm [9], continuous t-norm [15], residuated lattice [11-13]) is developed [8,10]. On the other hand, noncommutative structures play an important role in metric spaces, algebraic structures (groups, rings, quantales, pseudo-BL-algebras)[2-7]. Georgescu and Iorgulescu [5] introduced pseudo MV-algebras as the generalization of the MV-algebras. Georgescu and Popescu [6] introduced generalized residuated lattice as a noncommutative structure.

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In this paper, we investigate solutions of various types of fuzzy relation equations $A_i \circ R = B_i$, $R \circ A_i = B_i$, $A_i \Rightarrow R = B_i$ and $A_i \to R = B_i$ in generalized residuated lattice. We give their examples.

2. Preliminaries

Definition 2.1. [6] A structure $(L, \vee, \wedge, \odot, \to, \Rightarrow, \top, \bot)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation, i.e.

$$a \odot b \leq c \text{ iff } a \leq b \to c \text{ iff } b \leq a \Rightarrow c.$$

Remark 2.2.[6] (1) A generalized residuated lattice is a residuated lattice ($\to = \Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice.

(3) Let $(L, \leq, \odot)$ be a quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee\{z \in L \mid z \odot x \leq y\},$$
$$x \Rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$$(x \odot y) \leq z \text{ iff } x \leq (y \to z) \text{ iff } y \leq (x \Rightarrow z).$$

Hence $(L, \vee, \wedge, \odot, \to, \Rightarrow, \top, \bot)$ is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation, that is, $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \to \bot$ and $a^* = a \Rightarrow \bot$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, \Rightarrow, \top, \bot)$ is a generalized residuated lattice.

Lemma 2.3.[6] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$ for $\to \in \{\to, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$ for $\to \in \{\to, \Rightarrow\}$.

(4) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$, for $\to \in \{\to, \Rightarrow\}$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$, for $\to \in \{\to, \Rightarrow\}$.

(6) $(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(7) $x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ and $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.

(8) $x \odot (x \Rightarrow y) \leq y$ and $(x \to y) \odot x \leq y$.

(9) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \to z) \odot (x \to y) \leq x \to z$.

(10) $(x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z)$ and $(x \to z) \leq (x \odot y) \to (z \odot y)$.

(11) $(x \Rightarrow y) \leq (y \Rightarrow z) \to (x \Rightarrow z)$ and $(x \to y) \leq (y \to z) \Rightarrow (x \to z)$.

(12) $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq (\bigwedge_{i\in\Gamma} x_i) \to (\bigwedge_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.

(13) $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq (\bigvee_{i\in\Gamma} x_i) \to (\bigvee_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.

(14) $x \to y = \top$ iff $x \leq y$ iff $x \Rightarrow y = \top$.

3. Solutions of Fuzzy Relation Equations in Generalized Residuated Lattices

Definition 3.1. Let $A_i \in L^U$, $R \in L^{U\times V}$ and $B_i \in L^V$. We define fuzzy relation equations as follows, for each $i \in \{1, ..., n\}$,

$$\begin{aligned}
\mathbf{(1)}\ \ (A_i \circ R)(v) &= \textstyle\bigvee_{u\in U}(A_i(u) \odot R(u,v)) \\
&= B_i(v), \\
\mathbf{(2)}\ \ (R \circ A_i)(v) &= \textstyle\bigvee_{u\in U}(R(u,v) \odot A_i(u)) \\
&= B_i(v), \\
\mathbf{(3)}\ (A_i \Rightarrow R)(v) &= \textstyle\bigwedge_{v\in V}(A_i(u) \Rightarrow R(u,v)) \\
&= B_i(v), \\
\mathbf{(4)}\ (A_i \to R)(v) &= \textstyle\bigwedge_{v\in V}(A_i(u) \to R(u,v)) \\
&= B_i(v).
\end{aligned}$$

Let $U = \{u_1, ..., u_n\}$ be a set, $R \in L^{U\times V}$ an unknown fuzzy relation $A_1, ..., A_n \in L^U$ and $B_1, ..., B_n \in L^V$. If $v \in V$, $A_i(u_j) = a_{ij}$ for $i, j \in \{1, ..., n\}$, $R(u_j, v) = r_j$, $B_j(v) = b_j$, then the system **(1)** can be written by

$$\begin{array}{ll} a_{11} \odot r_1 \vee ... \vee a_{1n} \odot r_n & = b_1 \\ \vdots \quad \vdots & \vdots \\ a_{n1} \odot r_1 \vee ... \vee a_{nn} \odot r_n & = b_n. \end{array} \qquad (\mathbf{5})$$

The system **(2)** can be written by

$$\begin{array}{ll} r_1 \odot a_{11} \vee ... \vee r_n \odot a_{1n} & = b_1 \\ \vdots \quad \vdots & \vdots \\ r_1 \odot a_{n1} \vee ... \vee r_n \odot a_{nn} & = b_n. \end{array} \qquad (\mathbf{6})$$

Then system **(3)** can be written by

$$\begin{array}{ll} a_{11} \Rightarrow r_1 \wedge ... \wedge a_{1n} \Rightarrow r_n & = b_1 \\ \vdots \quad \vdots & \vdots \\ a_{n1} \Rightarrow r_1 \wedge ... \wedge a_{nn} \Rightarrow r_n & = b_n. \end{array} \qquad (\mathbf{7})$$

The system **(4)** can be written by

$$\begin{array}{ll} a_{11} \rightarrow r_1 \wedge ... \wedge a_{1n} \rightarrow r_n & = b_1 \\ \vdots \quad \vdots & \vdots \\ a_{n1} \rightarrow r_1 \wedge ... \wedge a_{nn} \rightarrow r_n & = b_n. \end{array} \qquad (\mathbf{8})$$

Theorem 3.2. (1) (5) is solvable iff $r^{\Rightarrow} = (r_1^{\Rightarrow}, ..., r_n^{\Rightarrow})$ with $r_j^{\Rightarrow} = \bigwedge_{i=1}^n (a_{ij} \Rightarrow b_i)$ for $j \in \{1, ..., n\}$ is the greatest solution.

(2) (6) is solvable iff $r^{\rightarrow} = (r_1^{\rightarrow}, ..., r_n^{\rightarrow})$ with $r_j^{\rightarrow} = \bigwedge_{i=1}^n (a_{ij} \rightarrow b_i)$ for $j \in \{1, ..., n\}$ is the greatest solution.

(3) (7) is solvable iff $r^{\odot} = (r_1^{\odot}, ..., r_n^{\odot})$ with $r_k^{\odot} = \bigvee_{i=1}^n (a_{ik} \odot b_i)$ is the least solution.

(4) (8) is solvable iff ${}^{\odot}r = ({}^{\odot}r_1, ..., {}^{\odot}r_n)$ with ${}^{\odot}r_k = \bigvee_{i=1}^{n}(b_i \odot a_{ik})$ is the least solution.

Proof. (1) ($\Rightarrow$) Let $r = (r_1, ..., r_n)$ be a solution of (5). Since $a_{ik} \odot r_k \leq \bigvee_{k=1}^{n}(a_{ik} \odot r_k) = b_i$, $1 \leq i \leq n$, then $r_k \leq a_{ik} \Rightarrow b_i$. Hence $r_k \leq \bigwedge_{i=1}^{n}(a_{ik} \Rightarrow b_i)$. Put $r_j^{\Rightarrow} = \bigwedge_{i=1}^{n}(a_{ij} \Rightarrow b_i)$. So,

$$\begin{aligned} b_i &= \bigvee_{k=1}^{n}(a_{ik} \odot r_k) \leq \bigvee_{k=1}^{n}(a_{ik} \odot \bigwedge_{i=1}^{n}(a_{ik} \Rightarrow b_i)) \\ &= \bigvee_{k=1}^{n}(a_{ik} \odot r_k^{\Rightarrow}) \leq \bigvee_{j=1}^{n}(a_{ij} \odot (a_{ij} \Rightarrow b_i)) \leq b_i. \end{aligned}$$

Thus, $\bigvee_{k=1}^{n}(a_{ik} \odot r_k^{\Rightarrow}) = b_i$. Hence $r^{\Rightarrow} = (r_1^{\Rightarrow}, ..., r_n^{\Rightarrow})$ with $r_j^{\Rightarrow} = \bigwedge_{i=1}^{n}(a_{ij} \Rightarrow b_i)$ is the greatest solution.

(2) ($\Rightarrow$) Let $r = (r_1, ..., r_n)$ be a solution of (6). Since $r_k \odot a_{ik} \leq \bigvee_{k=1}^{n}(r_k \odot a_{ik}) = b_i$, $1 \leq i \leq n$, then $r_k \leq a_{ik} \rightarrow b_i$. Hence $r_k \leq \bigwedge_{i=1}^{n}(a_{ik} \rightarrow b_i)$. Put $r_j^{\rightarrow} = \bigwedge_{i=1}^{n}(a_{ij} \rightarrow b_i)$. So,

$$\begin{aligned} b_i &= \bigvee_{k=1}^{n}(r_k \odot a_{ik}) \leq \bigvee_{k=1}^{n}(\bigwedge_{i=1}^{n}(a_{ik} \rightarrow b_i) \odot a_{ik}) \\ &= \bigvee_{k=1}^{n}(r_k^{\rightarrow} \odot a_{ik}) \leq \bigvee_{j=1}^{n}((a_{ij} \rightarrow b_i) \odot a_{ij}) \leq b_i. \end{aligned}$$

Thus, $\bigvee_{k=1}^{n}(r_k^{\rightarrow} \odot a_{ik}) = b_i$. Hence $r^{\rightarrow} = (r_1^{\rightarrow}, ..., r_n^{\rightarrow})$ with $r_j^{\rightarrow} = \bigwedge_{i=1}^{n}(a_{ij} \rightarrow b_i)$ is the greatest solution.

(3) Let $r = (r_1, ..., r_n)$ be a solution of (7). Since $b_i = \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k) \leq a_{ik} \Rightarrow r_k$, then $r_k \geq a_{ik} \odot b_i$. Thus $r_k \geq \bigvee_{i=1}^{n}(a_{ik} \odot b_i)$. Put $r_k^{\odot} = \bigvee_{i=1}^{n}(a_{ik} \odot b_i)$.

$$\begin{aligned} b_i &\leq \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow a_{ik} \odot b_i)) \\ &\leq \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow \bigvee_{p=1}^{m}(a_{pk} \odot b_p)) \\ &= \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k^{\odot}) \\ &\leq \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k) = b_i. \end{aligned}$$

Thus, $\bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k^{\odot}) = b_i$. Hence $r^{\odot} = (r_1^{\odot}, ..., r_n^{\odot})$ with $r_k^{\odot} = \bigvee_{k}^{n}(a_{ik} \odot b_i)$ is the least solution.

(4) Let $r = (r_1, ..., r_n)$ be a solution of (8). Since $b_i = \bigwedge_{k=1}^{n}(a_{ik} \rightarrow r_k) \leq a_{ik} \rightarrow r_k$, then $r_k \geq b_i \odot a_{ik}$. Thus $r_k \geq \bigvee_{i=1}^{n}(b_i \odot a_{ik})$.

$$\begin{aligned} b_i &\leq \bigwedge_{k=1}^{n}(a_{ik} \rightarrow (b_i \odot a_{ik})) \\ &\leq \bigwedge_{k=1}^{n}(a_{ik} \rightarrow \bigvee_{p=1}^{n}(b_p \odot a_{pk})) \\ &= \bigwedge_{k=1}^{n}(a_{ik} \rightarrow {}^{\odot} r_k) \\ &\leq \bigwedge_{k=1}^{n}(a_{ik} \rightarrow r_k) = b_i. \end{aligned}$$

Thus, $\bigwedge_{k=1}^{n}(a_{ik} \rightarrow {}^{\odot} r_k) = b_i$. Hence ${}^{\odot}r = ({}^{\odot}r_1, ..., {}^{\odot}r_n)$ with ${}^{\odot}r_k = \bigvee_{i=1}^{n}(b_i \odot a_{ik})$ is the least solution.

Theorem 3.3. (1) If (5) is solvable, then $\bigwedge_{k=1}^{n}(a_{ik} \to a_{jk}) \leq b_i \to b_j$.

(2) If (6) is solvable, then $\bigwedge_{k=1}^{n}(a_{ik} \Rightarrow a_{jk}) \leq b_i \Rightarrow b_j$.

(3) If (7) is solvable, then $\bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik}) \leq b_i \to b_j$.

(4) If (8) is solvable, then $\bigwedge_{k=1}^{n}(a_{jk} \to a_{ik}) \leq b_i \Rightarrow b_j$.

Proof. (1) Let $r = (r_1, ..., r_n)$ be a solution of (5). Since $\bigvee_{k=1}^{n}(a_{ik} \odot r_k) = b_i$, by Lemma 2.3 (10,13), we have

$$\begin{aligned} b_i \to b_j &= \bigvee_{k=1}^{n}(a_{ik} \odot r_k) \to \bigvee_{k=1}^{n}(a_{jk} \odot r_k) \\ &\geq \bigwedge_{k=1}^{n}((a_{ik} \odot r_k) \to (a_{jk} \odot r_k)) \\ &\geq \bigwedge_{k=1}^{n}(a_{ik} \to a_{jk}). \end{aligned}$$

(2) Let $r = (r_1, ..., r_n)$ be a solution of (6). Since $\bigvee_{k=1}^{n}(r_k \odot a_{ik}) = b_i$, by Lemma 2.3 (10,13), we have

$$\begin{aligned} b_i \Rightarrow b_j &= \bigvee_{k=1}^{n}(r_k \odot a_{ik}) \Rightarrow \bigvee_{k=1}^{n}(r_k \odot a_{jk}) \\ &\geq \bigwedge_{k=1}^{n}((r_k \odot a_{ik}) \Rightarrow (r_k \odot a_{jk})) \\ &\geq \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow a_{jk}). \end{aligned}$$

(3) Let $r = (r_1, ..., r_n)$ be a solution of (7). Since $\bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k) = b_i$, by Lemma 2.3 (11,12), we have

$$\begin{aligned} b_i \to b_j &= \bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k) \to \bigwedge_{k=1}^{n}(a_{jk} \Rightarrow r_k) \\ &\geq \bigwedge_{k=1}^{n}((a_{ik} \Rightarrow r_k) \to (a_{jk} \Rightarrow r_k)) \\ &\geq \bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik}). \end{aligned}$$

(4) Let $r = (r_1, ..., r_n)$ be a solution of (8). Since $\bigwedge_{k=1}^{n}(a_{jk} \to r_k) = b_j$, by Lemma 2.3 (11,12), we have

$$\begin{aligned} b_i \Rightarrow b_j &= \bigwedge_{k=1}^{n}(a_{ik} \to r_k) \Rightarrow \bigwedge_{k=1}^{n}(a_{jk} \to r_k) \\ &\geq \bigwedge_{k=1}^{n}((a_{ik} \to r_k) \Rightarrow (a_{jk} \to r_k)) \\ &\geq \bigwedge_{k=1}^{n}(a_{jk} \to a_{ik}). \end{aligned}$$

Definition 3.4. Let U be a set. A function $P : U \times U \to L$ is called a *right* $\odot$*-preorder* on U if it satisfies the following conditions:

(R) (reflexive) $P(u, u) = \top$ for all $u \in U$,

(RT) (right transitive) $P(u, v) \odot P(v, w) \leq P(u, w)$, for all $u, v, w \in U$.

A function $P : U \times U \to L$ is called a *left* $\odot$*-preorder* on U if it satisfies (R) and the following condition:

(LT) (left transitive) $P(v,w) \odot P(u,v) \le P(u,w)$, for all $u,v,w \in U$.

Remark 3.5. (1) If $P : U \times U \to L$ is a right (resp. left) $\odot$-preorder and define $P^t(u,v) = P(v,u)$, then $P^t : U \times U \to L$ is a left (resp. right) $\odot$-preorder.

(2) For $P \in L^{U\times U}$, we define $(P \circ P)(u,w) = \bigvee_{v\in U}(P(u,v) \odot P(v,w))$. Then P is right transitive iff $P \circ P \le P$. In particular, P is left transitive iff $P^t \circ P^t \le P^t$.

(3) Let U be a set. Define $P_1, P_2 : L^U \times L^U \to L$ as follows:

$$P_1(A,B) = \bigwedge_{u\in U} (A(u) \Rightarrow B(u)),$$

$$P_2(A,B) = \bigwedge_{u\in U} (A(u) \to B(u)).$$

By Lemma 2.3(9), P_1 is a right $\odot$-preorder on L^U and P_2 is a left $\odot$-preorder on L^U.

A fuzzy set $A \in L^X$ is call normal if there exists $u \in U$ such that $A(u) = \top$.

Theorem 3.6. Let $U = \{u_1, ..., u_n\}$ be a set and $A_i \in L^U$ normal for all $i \in \{1, ..., n\}$ such that $A_i(u_i) = \top$. Then the following statements are equivalent.

(1) There exists a right $\odot$-preorder P such that $A_i(u) = P(u_i,u)$ for all $i \in \{1, ..., n\}, u \in U$.

(2) $A_i(u) = \bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u))$ for all $i \in \{1, ..., n\}$.

(3) $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \to A_i(u))$ for all $i,j \in \{1, ..., n\}$.

Proof. (1)$\Rightarrow$(2). Since $P(u_i,u) \odot P(u,v) \le P(u_i,v)$, then $P(u,v) \le P(u_i,u) \Rightarrow P(u_i,v) = A_i(u) \Rightarrow A_i(v)$. Hence $P(u,v) \le \bigwedge_{k=1}^{n}(A_k(u) \Rightarrow A_k(v))$. So, $A_i(u) = P(u_i,u) \le \bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u)) \le A_i(u_i) \Rightarrow A_i(u) = A_i(u)$. Thus $A_i(u) = \bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u))$ for all $i \in \{1, ..., n\}, u \in U$.

(2)$\Rightarrow$(3). Since

$$\begin{aligned}&\textstyle\bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u_j)) \odot \bigwedge_{k=1}^{n}(A_k(u_j) \Rightarrow A_k(u))\\ &\le \textstyle\bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u))\\ &\text{iff } A_i(u_j) \odot A_j(u) \le A_i(u),\end{aligned}$$

then $A_i(u_j) \le \bigwedge_{u\in U}(A_j(u) \to A_i(u)) \le A_j(u_j) \to A_i(u_j) = A_i(u_j)$ for all $i,j \in \{1, ..., n\}$. Thus, $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \to A_i(u))$.

(3)$\Rightarrow$(1). Put $P(u,v) = \bigwedge_{k=1}^{n}(A_k(u) \Rightarrow A_k(v))$. Then P is a right $\odot$-preorder. Thus, $P(u_i,u) = \bigwedge_{k=1}^{n}(A_k(u_i) \Rightarrow A_k(u)) \le A_i(u_i) \Rightarrow A_i(u) = A_i(u)$.

Since $A_i(u_j) = \bigwedge_{x\in X}(A_j(u) \to A_i(u)) \le A_j(u) \to A_i(u)$, then $A_j(u) \le A_i(u_j) \Rightarrow A_i(u)$. Exchange i and j. Then $A_i(u) \le A_j(u_i) \Rightarrow A_j(u)$. So, $A_i(u) \le \bigwedge_{k=1}^n (A_k(u_i) \Rightarrow A_k(u))$. Hence $A_i(u) = P(u_i, u)$.

Theorem 3.7. Let $U = \{u_1, ..., u_n\}$ be a set and $A_i \in L^U$ normal for $i \in \{1, ..., n\}$ such that $A_i(u_i) = \top$. Then the following statements are equivalent.

(1) There exists a left $\odot$-preorder P such that $A_i(u) = P(u_i, u)$ for all $i \in \{1, ..., n\}, u \in U$.

(2) $A_i(u) = \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u))$ for all $i \in \{1, ..., n\}, u \in U$.

(3) $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u))$ for all $i, j \in \{1, ..., n\}$.

Proof. (1)$\Rightarrow$(2). Since P is a left $\odot$-preorder, then $P(u, v) \odot P(u_i, u) \le P(u_i, v)$. Thus, $P(u, v) \le P(u_i, u) \to P(u_i, v) = A_i(u) \to A_i(v)$. Hence $P(u, v) \le \bigwedge_{k=1}^n (A_k(u) \to A_k(v))$. So, $A_i(u) = P(u_i, u) \le \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u)) \le A_i(u_i) \to A_i(u) = A_i(u)$. Thus $A_i(u) = \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u))$ for all $i \in \{1, ..., n\}, u \in U$.

(2)$\Rightarrow$(3). Since

$$\begin{array}{l}\bigwedge_{k=1}^n (A_k(u_j) \to A_k(u)) \odot \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u_j)) \\ \le \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u)) \\ \text{iff } A_j(u) \odot A_i(u_j) \le A_i(u),\end{array}$$

then $A_i(u_j) \le \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u)) \le A_j(u_j) \Rightarrow A_i(u_j) = A_i(u_j)$ for all $i, j \in \{1, ..., n\}$. Thus, $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u))$.

(3)$\Rightarrow$(1). Put $P(u, v) = \bigwedge_{k=1}^n (A_k(u) \to A_k(v))$. Then P is a left $\odot$-preorder. So, $P(u_i, u) = \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u)) \le A_i(u_i) \to A_i(u) = A_i(u)$.

Since $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u)) \le A_j(u) \Rightarrow A_i(u)$, then $A_j(u) \le A_i(u_j) \to A_i(u)$. Exchange i and j. Then $A_i(u) \le \bigwedge_{k=1}^n (A_k(u_i) \to A_k(u)$. Hence $A_i(u) = P(u_i, u)$.

From Theorems 3.6 and 3.7, we obtain the following corollary.

Corollary 3.8. Let $U = \{u_1, ..., u_n\}$ be a set. Then the following statements hold.

(1) If $P \in L^{U\times U}$ is a right $\odot$-preorder and we put $A_i(u) = P(u_i, u) \in L^U$, then $A_i(u_i) = a_{ii} = \top$ and for all $i \in \{1, ..., n\}$,

$$A_i(u_j) = a_{ij} = \bigwedge_{k=1}^{n} (a_{ki} \Rightarrow a_{kj}) = \bigwedge_{k=1}^{n} (a_{jk} \to a_{ik}).$$

(2) If $P \in L^{U\times U}$ is a left $\odot$-preorder and we put $A_i(u) = P(u_i, u) \in L^U$, then $A_i(u_i) = a_{ii} = \top$ and for all $i \in \{1, ..., n\}$,

$$A_i(u_j) = a_{ij} = \bigwedge_{k=1}^{n} (a_{ki} \to a_{kj}) = \bigwedge_{k=1}^{n} (a_{jk} \Rightarrow a_{ik}).$$

Theorem 3.9. Let $A_i \in L^U$ be normal for $1 \le i \le n$ such that $a_{ii} = \top$ and $a_{ij} \le \bigwedge_{k=1}^{n}(a_{jk} \to a_{ik})$ for all $i, j \in \{1, ..., n\}$. Then the following statements hold.

(1) $\bigvee_{j=1}^{n}(a_{ij} \odot r_j) = b_i$ is solvable iff $a_{ij} \le b_j \to b_i$ for all $i, j \in \{1, ..., n\}$.

(2) $\bigwedge_{j=1}^{n}(a_{ij} \to r_j) = b_i$ is solvable iff $a_{ij} \le b_i \Rightarrow b_j$ for all $i, j \in \{1, ..., n\}$.

Proof. (1) ($\Rightarrow$) Let $\bigvee_{j=1}^{n}(a_{ij} \odot r_j) = b_i$ be solvable. Then $r_j^{\Rightarrow} = \bigwedge_{k=1}^{n}(a_{kj} \Rightarrow b_k)$ is a solution. So, $\bigvee_{j=1}^{n}(a_{ij} \odot r_j^{\Rightarrow}) = b_i$. Since $a_{ij} \le a_{jk} \to a_{ik}$ iff $a_{jk} \le a_{ij} \Rightarrow a_{ik}$, then

$$\bigvee_{k=1}^{n} ((a_{ij} \Rightarrow a_{ik}) \odot r_k^{\Rightarrow}) \ge \bigvee_{k=1}^{n} (a_{jk} \odot r_k^{\Rightarrow}) = b_j$$

Since $r_k^{\Rightarrow} \le a_{ik} \Rightarrow b_i$,

$$\begin{aligned} a_{ij} \Rightarrow b_i \ & \ge \textstyle\bigvee_{k=1}^{n}((a_{ij} \Rightarrow a_{ik}) \odot (a_{ik} \Rightarrow b_i)) \\ & \ge \textstyle\bigvee_{k=1}^{n}((a_{ij} \Rightarrow a_{ik}) \odot r_k^{\Rightarrow}) \ge b_j. \end{aligned}$$

Since $a_{ij} \Rightarrow b_i \ge b_j$ iff $a_{ij} \le b_j \to b_i$, then $a_{ij} \le b_j \to b_i$.

($\Leftarrow$) $r_j^{\Rightarrow} = \bigwedge_{k=1}^{n}(a_{kj} \Rightarrow b_k)$ is a solution from the following statement:

$$\bigvee_{j=1}^{n} (a_{ij} \odot r_j^{\Rightarrow}) \le \bigvee_{j=1}^{n} (a_{ij} \odot (a_{ij} \Rightarrow b_i)) \le b_i,$$

$$\begin{aligned} & \textstyle\bigvee_{j=1}^{n}(a_{ij} \odot r_j^{\Rightarrow}) \ge a_{ii} \odot r_i^{\Rightarrow} = \top \odot r_i^{\Rightarrow} \\ & = \textstyle\bigwedge_{j=1}^{n}(a_{ji} \Rightarrow b_j) \ge b_i \end{aligned}$$

because $a_{ji} \le b_i \to b_j$ iff $b_i \le a_{ji} \Rightarrow b_j$.

(2) ($\Rightarrow$) Let $\bigwedge_{j=1}^{n}(a_{ij} \to r_j) = b_i$ be solvable. Then ${}^{\odot}r_k = \bigvee_{j=1}^{n}(b_j \odot a_{jk})$ is a solution. That is, $\bigwedge_{k=1}^{n}(a_{ik} \to^{\odot} r_k) = b_i$. Thus

$$\bigwedge_{k=1}^{n} (a_{jk} \to b_i \odot a_{ik}) \le b_j.$$

Since $a_{ij} \le a_{jk} \to a_{ik}$,

$$b_i \odot a_{ij} \odot a_{jk} \le b_i \odot (a_{jk} \to a_{ik}) \odot a_{jk} \le b_i \odot a_{ik}$$

Thus $b_i \odot a_{ij} \leq a_{jk} \to b_i \odot a_{ik}$. So, $b_i \odot a_{ij} \leq \bigwedge_{k=1}^n (a_{jk} \to b_i \odot a_{ik}) \leq b_j$. Thus, $a_{ij} \leq b_i \Rightarrow b_j$.

($\Leftarrow$) We will show that ${}^{\odot}r_k = \bigvee_{j=1}^n (b_j \odot a_{jk})$ is a solution.

$$\bigwedge_{k=1}^n (a_{ik} \to^{\odot} r_k) \geq \bigwedge_{k=1}^n (a_{ik} \to b_i \odot a_{ik}) \geq b_i.$$

On the other hand, since $a_{ij} \leq b_i \Rightarrow b_j$, we have

$$\begin{aligned} &\bigwedge_{k=1}^n (a_{ik} \to^{\odot} r_k) \leq a_{ii} \to^{\odot} r_i = \top \to^{\odot} r_i = ^{\odot} r_i \\ &= \bigvee_{j=1}^n (b_j \odot a_{ji}) \leq \bigvee_{j=1}^n (b_j \odot (b_j \Rightarrow b_i)) \leq b_i. \end{aligned}$$

From Theorem 3.9, we obtain the following corollary.

Corollary 3.10. Let $U = \{u_1, ..., u_n\}$ be a set and $P \in L^{U\times U}$ a right $\odot$-preorder with $A_i(u) = P(u_i, u) \in L^U$. Then the following statements hold.

(1) $\bigvee_{j=1}^n (a_{ij} \odot r_j) = b_i$ is solvable iff $a_{ij} = \bigwedge_{k=1}^n (a_{ki} \Rightarrow a_{kj}) = \bigwedge_{k=1}^n (a_{jk} \to a_{ik}) \leq b_j \to b_i$ for all $i, j \in \{1, ..., n\}$.

(2) $\bigwedge_{j=1}^n (a_{ij} \to r_j) = b_i$ is solvable iff $a_{ij} = \bigwedge_{k=1}^n (a_{ki} \Rightarrow a_{kj}) = \bigwedge_{k=1}^n (a_{jk} \to a_{ik}) \leq b_i \Rightarrow b_j$ for all $i, j \in \{1, ..., n\}$.

Theorem 3.11. Let $A_i \in L^U$ be normal for $1 \leq i \leq n$ such that $a_{ii} = \top$ and $a_{ij} \leq \bigwedge_{k=1}^n (a_{jk} \Rightarrow a_{ik})$ for all $i, j \in I$. Then the following statements hold.

(1) $\bigvee_{j=1}^n (r_j \odot a_{ij}) = b_i$ is solvable iff $a_{ij} \leq b_j \Rightarrow b_i$ for all $i, j = \{1, ..., n\}$.

(2) $\bigwedge_{j=1}^n (a_{ij} \Rightarrow r_j) = b_i$ is solvable iff $a_{ij} \leq b_i \to b_j$ for all $i, j = \{1, ..., n\}$.

Proof. (1) ($\Rightarrow$) Let $\bigvee_{j=1}^n (r_j \odot a_{ij}) = b_i$ be solvable. Then $r_j^{\to} = \bigwedge_{k=1}^n (a_{kj} \to b_k)$ is a solution. So, $\bigvee_{j=1}^n (r_j^{\to} \odot a_{ij}) = b_i$. Since $a_{ij} \leq a_{jk} \Rightarrow a_{ik}$ iff $a_{jk} \leq a_{ij} \to a_{ik}$, then

$$\bigvee_{k=1}^n (r_k^{\to} \odot (a_{ij} \to a_{ik})) \geq \bigvee_{k=1}^n (r_k^{\to} \odot a_{jk}) = b_j.$$

Since $r_k^{\to} \leq a_{ik} \to b_i$,

$$\begin{aligned} a_{ij} \to b_i \quad &\geq \bigvee_{k=1}^n ((a_{ik} \to b_i) \odot (a_{ij} \to a_{ik}) \\ &\geq \bigvee_{k=1}^n (r_k^{\to} \odot (a_{ij} \to a_{ik})) \geq b_j. \end{aligned}$$

Since $a_{ij} \to b_i \geq b_j$ iff $a_{ij} \leq b_j \Rightarrow b_i$, then $a_{ij} \leq b_j \Rightarrow b_i$.

($\Leftarrow$) $r_j^{\to} = \bigwedge_{k=1}^n (a_{kj} \to b_k)$ is a solution from the following statement:

$$\bigvee_{j=1}^n (r_j^{\to} \odot a_{ij}) \leq \bigvee_{j=1}^n ((a_{ij} \to b_i) \odot a_{ij}) \leq b_i,$$

$$\begin{aligned} \bigvee_{j=1}^{n}(r_j^{\rightarrow} \odot a_{ij}) &\geq r_i^{\rightarrow} \odot a_{ii} = r_i^{\rightarrow} \odot \top \\ &= \bigwedge_{k=1}^{n}(a_{ki} \rightarrow b_k) \geq b_i \end{aligned}$$

because $a_{ji} \leq b_i \Rightarrow b_j$ iff $b_i \leq a_{ji} \rightarrow b_j$.

(2) ($\Rightarrow$) Let $\bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k) = b_i$ be solvable. Then $r_k^{\odot} = \bigvee_{j=1}^{n}(a_{jk} \odot b_j)$ is a solution. That is, $\bigwedge_{k=1}^{n}(a_{ik} \Rightarrow r_k^{\odot}) = b_i$. Thus

$$\bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik} \odot b_i) \leq b_j.$$

Since $a_{ij} \leq a_{jk} \Rightarrow a_{ik}$,

$$a_{jk} \odot a_{ij} \odot b_i \leq a_{jk} \odot (a_{jk} \Rightarrow a_{ik}) \odot b_i \leq a_{ik} \odot b_i$$

Thus $a_{ij} \odot b_i \leq a_{jk} \Rightarrow a_{ik} \odot b_i$. So, $a_{ij} \odot b_i \leq \bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik} \odot b_i) \leq b_j$.

($\Leftarrow$) We will show that ${}^{\odot}r_k = \bigvee_{j=1}^{n}(b_j \odot a_{jk})$ is a solution.

$$\bigwedge_{k=1}^{n}(a_{ik} \rightarrow^{\odot} r_k) \geq \bigwedge_{k=1}^{n}(a_{ik} \rightarrow b_i \odot a_{ik}) \geq b_i.$$

On the other hand, since $a_{ij} \leq b_i \Rightarrow b_j$, we have

$$\begin{aligned} &\bigwedge_{k=1}^{n}(a_{ik} \rightarrow^{\odot} r_k) \leq a_{ii} \rightarrow^{\odot} r_i = \top \rightarrow^{\odot} r_i =^{\odot} r_i \\ &= \bigvee_{j=1}^{n}(b_j \odot a_{ji}) \leq \bigvee_{j=1}^{n}(b_j \odot (b_j \Rightarrow b_i)) \leq b_i. \end{aligned}$$

Thus, $\bigwedge_{k=1}^{n}(a_{ik} \rightarrow^{\odot} r_k) = b_i$.

From Theorem 3.11, we obtain the following corollary.

Corollary 3.12. Let $U = \{u_1, ..., u_n\}$ be a set and $P \in L^{U \times U}$ a left $\odot$-preorder with $A_i(u) = P(u_i, u) \in L^U$. Then the following statements hold.

(1) $\bigvee_{j=1}^{n}(r_j \odot a_{ij}) = b_i$ is solvable iff $a_{ij} = \bigwedge_{k=1}^{n}(a_{ki} \rightarrow a_{kj}) = \bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik}) \leq b_j \Rightarrow b_i$ for all $i, j \in \{1, ..., n\}$.

(2) $\bigvee_{j=1}^{n}(a_{ij} \Rightarrow r_j) = b_i$ is solvable iff $a_{ij} = \bigwedge_{k=1}^{n}(a_{ki} \rightarrow a_{kj}) = \bigwedge_{k=1}^{n}(a_{jk} \Rightarrow a_{ik}) \leq b_i \rightarrow b_j$ for all $i, j \in \{1, ..., n\}$.

Theorem 3.13. Let $A_i \in L^U$ be normal for $1 \leq i \leq n$ such that $a_{ii} = \top$. Then the following properties hold.

(1) $R^{\odot} = (R_1^{\odot}, ..., R_n^{\odot})$ with $R_j^{\odot} = \bigvee_{i=1}^{n}(a_{ij} \odot b_i)$ is a solution of (5) iff $\bigvee_{k=1}^{n}(a_{ik} \odot a_{jk}) \leq (b_j \rightarrow b_i), \forall i, j \in \{1, ..., n\}$.

(2) ${}^{\odot}R = ({}^{\odot}R_1, ..., {}^{\odot}R_n)$ with ${}^{\odot}R_k = \bigvee_{i=1}^{n}(b_i \odot a_{ik})$ is a solution of (6) iff $\bigvee_{k=1}(a_{jk} \odot a_{ik}) \leq (b_j \Rightarrow b_i), \forall i, j \in \{1, ..., n\}$.

Proof. (1) Let $R^{\odot} = (R_1^{\odot}, ..., R_n^{\odot})$ with $R_j^{\odot} = \bigvee_{i=1}^n (a_{ij} \odot b_i)$ be a solution of (5). Then

$$\begin{aligned} b_i &= \bigvee_{k=1}^n (a_{ik} \odot R_k^{\odot}) = \bigvee_{k=1}^n (a_{ik} \odot \bigvee_{i=1}^n (a_{ik} \odot b_i)) \\ &\geq a_{ik} \odot a_{jk} \odot b_j. \end{aligned}$$

Hence $\bigvee_{k=1}^n (a_{ik} \odot a_{jk}) \leq (b_j \rightarrow b_i), \forall i, j.$

Conversely, since $A_i \in L^U$ is normal for $1 \leq i \leq n$, there exists $u_i \in U$ such that $a_{ii} = \top$. Hence $b_i = a_{ii} \odot a_{ii} \odot b_i \leq \bigvee_{k=1}^n (a_{ik} \odot R_k^{\odot})$. Since $a_{ik} \odot a_{jk} \leq b_j \rightarrow b_i$ iff $a_{ik} \odot a_{jk} \odot b_j \leq b_i$, then $\bigvee_{k=1}^n (a_{ik} \odot R_k^{\odot}) \leq b_i$. So, $\bigvee_{k=1}^n (a_{ik} \odot R_k^{\odot}) = b_i$. Hence $R^{\odot} = (R_1^{\odot}, ..., R_n^{\odot})$ with $R_k^{\odot} = \bigvee_{i=1}^n (a_{ik} \odot b_i)$ is a solution of (5).

(2) Let ${}^{\odot}R = ({}^{\odot}R_1, ..., {}^{\odot}R_n)$ with ${}^{\odot}R_k = \bigvee_{i=1}^n (b_i \odot a_{ik})$ is a solution of (6). Then

$$\begin{aligned} b_i &= \bigvee_{k=1}^n ({}^{\odot}R_k \odot a_{ik}) \\ &= \bigvee_{k=1}^n (\bigvee_{j=1}^n (b_j \odot a_{jk}) \odot a_{ik}) \\ &\geq b_j \odot a_{jk} \odot a_{ik}. \end{aligned}$$

Hence $\bigvee_{k=1}^n (a_{ik} \odot a_{jk}) \leq b_j \Rightarrow b_i, \forall i, j \in \{1, ..., n\}$.

Conversely, since $A_i \in L^U$ is normal for $1 \leq i \leq n$, there exists $u_i \in U$ such that $a_{ii} = \top$. Hence $b_i = b_i \odot a_{ii} \odot a_{ii} \leq \bigvee_{k=1}^n ({}^{\odot}R_k \circ a_{ik})$. Since $a_{jk} \odot a_{ik} \leq b_j \Rightarrow b_i$ iff $b_j \odot a_{jk} \odot a_{ik} \leq b_i$, then $\bigvee_{k=1}^n ({}^{\odot}R_k \circ a_{ik}) \leq b_i$. So, $\bigvee_{k=1}^n ({}^{\odot}R_k \circ a_{ik}) = b_i$. Hence ${}^{\odot}R = ({}^{\odot}R_1, ..., {}^{\odot}R_n)$ with ${}^{\odot}R_k = \bigvee_{i=1}^n (b_i \odot a_{ik})$ is a solution of (6).

Example 3.15. Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \rightarrow K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned} &(x_1, y_1) \leq (x_2, y_2) \\ &\Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P, \\ &\ (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then $(K, \leq \otimes)$ is a lattice-group. (ref. [1])

The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is a generalized residuated lattice where $\perp = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element from the following statements:

$$\begin{aligned}(x_1,y_1)\odot(x_2,y_2) &= (x_1,y_1)\otimes(x_2,y_2)\vee(\tfrac{1}{2},1)\\ &= (x_1x_2, x_1y_2+y_1)\vee(\tfrac{1}{2},1),\\ (x_1,y_1)\Rightarrow(x_2,y_2) &= ((x_1,y_1)^{-1}\otimes(x_2,y_2))\wedge(1,0)\\ &= (\tfrac{x_2}{x_1},\tfrac{y_2-y_1}{x_1})\wedge(1,0),\\ (x_1,y_1)\rightarrow(x_2,y_2) &= ((x_2,y_2)\otimes(x_1,y_1)^{-1})\wedge(1,0)\\ &= (\tfrac{x_2}{x_1},-\tfrac{x_2y_1}{x_1}+y_2)\wedge(1,0).\end{aligned}$$

Let $U=\{u_1,u_2,u_3\}$ be a set. Define $P\in L^{U\times U}$ as

$$P=\begin{pmatrix}(1,0) & (\frac{5}{8},\frac{5}{2}) & (\frac{5}{6},\frac{5}{3})\\ (\frac{5}{7},\frac{30}{7}) & (1,0) & (\frac{5}{8},-\frac{5}{4})\\ (1,-2) & (\frac{5}{7},\frac{10}{3}) & (1,0)\end{pmatrix}$$

Since $P\circ P\le P$ and $P^t\circ P^t\le P^t$, by Remark 3.5(2), we easily show that P is a right $\odot$-preorder and P^t is a left $\odot$-preorder.

(1) Put $A_i=(A_i(u_1),A_i(u_2),A_i(u_3))$ for each $i=\{1,2,3\}$ as follows

$$A_1=((1,0),(\frac{5}{8},\frac{5}{2}),(\frac{5}{6},\frac{5}{3}))$$

$$A_2=((\frac{5}{7},\frac{30}{7}),(1,0),(\frac{5}{8},-\frac{5}{4}))$$

$$A_3=((1,-2),(\frac{5}{7},\frac{10}{3}),(1,0))$$

$$b=(b_1,b_2,b_3)=((\frac{3}{5},1),(\frac{5}{6},-3),(\frac{2}{3},2))$$

(1-1)

$$\begin{aligned}&(1,0)\odot r_1\vee(\tfrac{5}{8},\tfrac{5}{2})\odot r_2\vee(\tfrac{5}{6},\tfrac{5}{3})\odot r_3=(\tfrac{3}{5},1)\\ &(\tfrac{5}{7},\tfrac{30}{7})\odot r_1\vee(1,0)\odot r_2\vee(\tfrac{5}{8},-\tfrac{5}{4})\odot r_3=(\tfrac{5}{6},-3)\\ &(1,-2)\odot r_1\vee(\tfrac{5}{7},\tfrac{10}{3})\odot r_2\vee(1,0)\odot r_3=(\tfrac{2}{3},2)\end{aligned}\qquad (\mathbf{9})$$

Since $P=(a_{ij})$ is a right $\odot$-preorder, by Corollary 3.8(1), $A_i(u)=P(u_i,u)$ and $A_i(u_j)=\bigwedge_{u\in U}(A_j(u)\rightarrow A_i(u))$;i.e. $a_{ij}=\bigwedge_{k=1}^{3}(a_{jk}\rightarrow a_{ik})$. Since $a_{ij}\le b_j\rightarrow b_i$ for all $i,j=\{1,2,3\}$, by Corollary 3.10(1), $\bigvee_{j=1}^{3}(a_{ij}\odot r_j)=b_i$ is solvable. By Theorem 3.2, $r^{\Rightarrow}=((\frac{3}{5},1),(\frac{5}{6},-3),(\frac{2}{3},2))$ with $r_j^{\Rightarrow}=\bigwedge_{k=1}^{3}(a_{kj}\Rightarrow b_k)$ is a solution.

Since $(1,-2)=\bigvee_{j=1}^{3}(a_{1j}\odot a_{3j})\not\le(b_3\rightarrow b_1)=(\frac{9}{10},-\frac{4}{5})$, by Theorem 3.13 (1), $R^{\odot}=((\frac{2}{3},0),(\frac{5}{6},-3),(\frac{2}{3},2))$ with $R_k^{\odot}=\bigvee_{i=1}^{n}(a_{ik}\odot b_i)$ is not a solution of (9).

(1-2)

$$\begin{array}{l} r_1 \odot (1,0) \vee r_2 \odot (\frac{5}{8},\frac{5}{2}) \vee r_3 \odot (\frac{5}{6},\frac{5}{3}) = (\frac{3}{5},1)(\frac{2}{3},2) \\ r_1 \odot (\frac{5}{7},\frac{30}{7}) \vee r_2 \odot (1,0) \vee r_3 \odot (\frac{5}{8},-\frac{5}{4}) = (\frac{5}{6},-3) \\ r_1 \odot (1,-2) \vee r_2 \odot (\frac{5}{7},\frac{10}{3}) \vee r_3 \odot (1,0) = (\frac{2}{3},2) \end{array} \quad (\mathbf{10})$$

Since $P = (a_{ij})$ is a left $\odot$-preorder, by Corollary 3.8(2), $A_i(u) = P(u_i, u)$ and $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u))$;i.e. $a_{ij} = \bigwedge_{k=1}^{3}(a_{jk} \Rightarrow a_{ik})$. Since $a_{ij} \leq b_j \Rightarrow b_i$ for all $i,j = \{1,2,3\}$, by Corollary 3.12(1), $\bigvee_{k=1}^{3}(r_k \odot a_{ik}) = b_i$ is solvable. Then $r^{\rightarrow} = ((\frac{3}{5},1),(\frac{5}{6},-3),(\frac{2}{3},2))$ with $r_k^{\rightarrow} = \bigwedge_{j=1}^{3}(a_{jk} \rightarrow b_j)$ is a solution of (10). Since $(1,-2) = \bigvee_{j=1}^{3}(a_{3j} \odot a_{1j}) \not\leq (b_3 \Rightarrow b_1) = (\frac{9}{10},-\frac{3}{2})$, by Theorem 3.13(2), ${}^{\odot}R = ((\frac{2}{3},\frac{2}{3}),(\frac{5}{6},-3),(\frac{2}{3},2))$ with ${}^{\odot}R_k = \bigvee_{i=1}^{3}(b_i \odot a_{ik})$ is not a solution of (10).

(1-3)

$$\begin{array}{l} (1,0) \rightarrow r_1 \wedge (\frac{5}{8},\frac{5}{2}) \rightarrow r_2 \wedge (\frac{5}{6},\frac{5}{3}) \rightarrow r_3 = (\frac{3}{5},1) \\ (\frac{5}{7},\frac{30}{7}) \rightarrow r_1 \wedge (1,0) \rightarrow r_2 \wedge (\frac{5}{8},-\frac{5}{4}) \rightarrow r_3 = (\frac{5}{6},-3) \\ (1,-2) \rightarrow r_1 \wedge (\frac{5}{7},\frac{10}{3}) \rightarrow r_2 \wedge (1,0) \rightarrow r_3 = (\frac{2}{3},2) \end{array} \quad (\mathbf{11})$$

Since $P = (a_{ij})$ is a right $\odot$-preorder, by Corollary 3.8(1), $A_i(u) = P(u_i, u)$ and $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \rightarrow A_i(u))$;i.e. $a_{ij} = \bigwedge_{k=1}^{3}(a_{jk} \rightarrow a_{ik})$. Since $(1,-2) = a_{31} \not\leq b_1 \Rightarrow b_3 = (\frac{9}{10},-\frac{3}{2})$, by corollary 3.10(2), $\bigwedge_{k=1}^{3}(a_{ik} \rightarrow r_k) = b_i$ is not solvable. Thus ${}^{\odot}r = ((\frac{2}{3},-\frac{4}{3}),(\frac{5}{6},-3),(\frac{2}{3},0))$ with ${}^{\odot}r_k = \bigvee_{j=1}^{n}(b_j \odot a_{jk})$ is not a solution of (11).

(1-4)

$$\begin{array}{l} (1,0) \Rightarrow r_1 \wedge (\frac{5}{8},\frac{5}{2}) \Rightarrow r_2 \wedge (\frac{5}{6},\frac{5}{3}) \Rightarrow r_3 = (\frac{3}{5},1) \\ (\frac{5}{7},\frac{30}{7}) \Rightarrow r_1 \wedge (1,0) \Rightarrow r_2 \wedge (\frac{5}{8},-\frac{5}{4}) \Rightarrow r_3 = (\frac{5}{6},-3) \\ (1,-2) \Rightarrow r_1 \wedge (\frac{5}{7},\frac{10}{3}) \Rightarrow r_2 \wedge (1,0) \Rightarrow r_3 = (\frac{2}{3},2) \end{array} \quad (\mathbf{12})$$

Since $P = (a_{ij})$ is a left $\odot$-preorder, by Corollary 3.8(1), $A_i(u) = P(u_i, u)$ and $A_i(u_j) = \bigwedge_{u\in U}(A_j(u) \Rightarrow A_i(u))$;i.e. $a_{ij} = \bigwedge_{k=1}^{3}(a_{jk} \Rightarrow a_{ik})$. Since $(1,-2) = a_{31} \not\leq b_1 \rightarrow b_3 = (\frac{9}{10},-\frac{4}{5})$, by Corollary 3.12(2), $\bigwedge_{k=1}^{3}(a_{ik} \Rightarrow r_k) = b_i$ is not solvable. Thus ${}^{\odot}r = ((\frac{2}{3},-\frac{4}{3}),(\frac{5}{6},-3),(\frac{2}{3},0))$ with ${}^{\odot}r_k = \bigvee_{j=1}^{n}(a_{jk} \odot b_j)$ is not a solution of (12).

(2) Put $A_i \in L^U$ for $i \in \{1,2,3\}$ as same in (1) and

$$b = (b_1, b_2, b_3) = ((\frac{2}{3},2),(\frac{5}{6},-3),(\frac{3}{5},1)).$$

(2-1)

$$\begin{array}{l}(1,0)\to r_1\wedge(\frac{5}{8},\frac{5}{2})\to r_2\wedge(\frac{5}{6},\frac{5}{3})\to r_3=(\frac{2}{3},2)\\ (\frac{5}{7},\frac{30}{7})\to r_1\wedge(1,0)\to r_2\wedge(\frac{5}{8},-\frac{5}{4})\to r_3=(\frac{5}{6},-3)\\ (1,-2)\to r_1\wedge(\frac{5}{7},\frac{10}{3})\to r_2\wedge(1,0)\to r_3=(\frac{3}{5},1)\end{array}\qquad (\mathbf{13})$$

Since $P=(a_{ij})$ is a right $\odot$-preorder, by Corollary 3.8(1), $A_i(u)=P(u_i,u)$ and $A_i(u_j)=\bigwedge_{u\in U}(A_j(u)\to A_i(u))$;i.e. $a_{ij}=\bigwedge_{k=1}^3(a_{jk}\to a_{ik})$. Since $a_{ij}\le b_i\Rightarrow b_j$, by Corollary 3.10(2), $\bigwedge_{k=1}^3(a_{ik}\to r_k)=b_i$ is solvable. By Theorem 3.2(4). ${}^{\odot}r=((\frac{2}{3},2),(\frac{5}{6},-3),(\frac{3}{5},1))$ with ${}^{\odot}r_k=\bigvee_{j=1}^3(b_j\odot a_{jk})$ is a solution of (13).

(2-2)

$$\begin{array}{l}(1,0)\Rightarrow r_1\wedge(\frac{5}{8},\frac{5}{2})\Rightarrow r_2\wedge(\frac{5}{6},\frac{5}{3})\Rightarrow r_3=(\frac{2}{3},2)\\ (\frac{5}{7},\frac{30}{7})\Rightarrow r_1\wedge(1,0)\Rightarrow r_2\wedge(\frac{5}{8},-\frac{5}{4})\Rightarrow r_3=(\frac{5}{6},-3)\\ (1,-2)\Rightarrow r_1\wedge(\frac{5}{7},\frac{10}{3})\Rightarrow r_2\wedge(1,0)\Rightarrow r_3=(\frac{3}{5},1)\end{array}\qquad (\mathbf{14})$$

Since $P=(a_{ij})$ is a left $\odot$-preorder, by Corollary 3.8(2), $A_i(u)=P(u_i,u)$ and $A_i(u_j)=\bigwedge_{u\in U}(A_j(u)\Rightarrow A_i(u))$;i.e. $a_{ij}=\bigwedge_{k=1}^3(a_{jk}\Rightarrow a_{ik})$. Since $a_{ij}\le b_i\to b_j$, by Corollary 3.12(2), $\bigwedge_{k=1}^3(a_{ik}\Rightarrow r_k)=b_i$ is solvable. By Theorem 3.2(3), ${}^{\odot}r=((\frac{2}{3},2),(\frac{5}{6},-3),(\frac{3}{5},1))$ with $r_k^{\odot}=\bigvee_{j=1}^3(a_{jk}\odot b_j)$ is a solution of (14).

(2-3)

$$\begin{array}{l}(1,0)\odot r_1\vee(\frac{5}{8},\frac{5}{2})\odot r_2\vee(\frac{5}{6},\frac{5}{3})\odot r_3=(\frac{2}{3},2)\\ (\frac{5}{7},\frac{30}{7})\odot r_1\vee(1,0)\odot r_2\vee(\frac{5}{8},-\frac{5}{4})\odot r_3=(\frac{5}{6},-3)\\ (1,-2)\odot r_1\vee(\frac{5}{7},\frac{10}{3})\odot r_2\vee(1,0)\odot r_3=(\frac{3}{5},1)\end{array}\qquad (\mathbf{15})$$

Since $(1,-2)=a_{31}\not\le b_1\to b_3=(\frac{9}{10},-\frac{4}{5})$, by Corollary 3.10(1), $\bigvee_{k=1}^3(a_{ik}\odot r_k)=b_i$ is not solvable. Then $r_k^{\Rightarrow}=\bigwedge_{j=1}^3(a_{jk}\Rightarrow b_j)$ is not a solution such that $r^{\Rightarrow}=((\frac{3}{5},3),(\frac{5}{6},-3),(\frac{2}{3},2))$ ..

(2-4)

$$\begin{array}{l}r_1\odot(1,0)\vee r_2\odot(\frac{5}{8},\frac{5}{2})\vee r_3\odot(\frac{5}{6},\frac{5}{3})=(\frac{2}{3},2)\\ r_1\odot(\frac{5}{7},\frac{30}{7})\vee r_2\odot(1,0)\vee r_3\odot(\frac{5}{8},-\frac{5}{4})=(\frac{5}{6},-3)\\ r_1\odot(1,-2)\vee r_2\odot(\frac{5}{7},\frac{10}{3})\vee r_3\odot(1,0)=(\frac{3}{5},1)\end{array}\qquad (\mathbf{16})$$

Since $(1,-2)=a_{31}\not\le b_1\Rightarrow b_3=(\frac{9}{10},-\frac{3}{2})$, by Corollary 3.12(1), $\bigvee_{k=1}^3(r_k\odot a_{ik})=b_i$ is not solvable. Then $r^{\to}=((\frac{3}{5},\frac{11}{5}),(\frac{5}{6},-3),(\frac{2}{3},2))$ with $r_k^{\to}=\bigwedge_{j=1}^3(a_{jk}\to b_j)$ is not a solution of (16).

(3) Put $A_i=(A_i(u_1),A_i(u_2),A_i(u_3))$ for each $i=\{1,2,3\}$ as follows

$$A_1=((1,0),(\frac{5}{8},\frac{5}{2}),(\frac{3}{5},\frac{5}{3}))$$

$$A_2 = ((\frac{5}{7}, \frac{30}{7}), (1,0), (\frac{5}{8}, -\frac{5}{4}))$$

$$A_3 = ((\frac{2}{3}, -2), (\frac{5}{7}, \frac{10}{3}), (1,0))$$

$$b = (b_1, b_2, b_3) = ((\frac{5}{6}, -3), (\frac{3}{4}, 1), (1, -2))$$

(3-1)

$$\begin{array}{l} (1,0) \odot r_1 \vee (\frac{5}{8}, \frac{5}{2}) \odot r_2 \vee (\frac{3}{5}, \frac{5}{3}) \odot r_3 = (\frac{5}{6}, -3) \\ (\frac{5}{7}, \frac{30}{7}) \odot r_1 \vee (1,0) \odot r_2 \vee (\frac{5}{8}, -\frac{5}{4}) \odot r_3 = (\frac{3}{4}, 1) \\ (\frac{2}{3}, -2) \odot r_1 \vee (\frac{5}{7}, \frac{10}{3}) \odot r_2 \vee (1,0) \odot r_3 = (1, -2) \end{array} \quad (\mathbf{17})$$

Put $P = (a_{ij})$. Since $P \circ P \leq P$ and $P^t \circ P^t \leq P^t$, by Remark 3.5(2), P is a right $\odot$-preorder and P is a left $\odot$-preorder. Since $a_{ij} \leq b_j \to b_i$ for all $i, j = \{1, 2, 3\}$, by Corollary 3.10(1), $\bigvee_{k=1}^{3}(a_{ik} \odot r_k) = b_i$ is solvable. From Theorem 3.2(1), $r^{\Rightarrow} = ((\frac{5}{6}, -3), (\frac{3}{4}, 1), (1, -2))$ with $r_k^{\Rightarrow} = \bigwedge_{j=1}^{3}(a_{jk} \Rightarrow b_j)$ is a solution.

Since $\bigvee_{k=1}^{3}(a_{ik} \odot a_{jk}) \leq (b_j \to b_i)$, by Theorem 3.13 (1), $R^{\odot} = ((\frac{5}{6}, -3), (\frac{3}{4}, 1), (1, -2))$ with $R_k^{\odot} = \bigvee_{i=1}^{3}(a_{ik} \odot b_i)$ is a solution of (17).

(3-2)

$$\begin{array}{l} r_1 \odot (1,0) \vee r_2 \odot (\frac{5}{8}, \frac{5}{2}) \vee r_3 \odot (\frac{3}{5}, \frac{5}{3}) = (\frac{5}{6}, -3) \\ r_1 \odot (\frac{5}{7}, \frac{30}{7}) \vee r_2 \odot (1,0) \vee r_3 \odot (\frac{5}{8}, -\frac{5}{4}) = (\frac{3}{4}, 1) \\ r_1 \odot (\frac{2}{3}, -2) \vee r_2 \odot (\frac{5}{7}, \frac{10}{3}) \vee r_3 \odot (1,0) = (1, -2) \end{array} \quad (\mathbf{18})$$

Since $a_{ij} \leq b_j \Rightarrow b_i$ for all $i, j = \{1, 2, 3\}$, by Corollary 3.10(2), $\bigvee_{k=1}^{3}(r_k \odot a_{ik}) = b_i$ is solvable. Then $r^{\rightarrow} = ((\frac{5}{6}, -3), (\frac{3}{4}, 1), (1, -2))$ with $r_k^{\rightarrow} = \bigwedge_{j=1}^{3}(a_{jk} \to b_j)$ is a solution. Since $\bigvee_{k=1}^{3}(a_{ik} \odot a_{jk}) \leq (b_i \Rightarrow b_j)$, by Theorem 3.13 (2), ${}^{\odot}R = ((\frac{5}{6}, -3), (\frac{3}{4}, 1), (1, -2))$ with ${}^{\odot}R_k = \bigvee_{i=1}^{n}(b_i \odot a_{ik})$ is a solution of (18).

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