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COMMON FIXED POINT THEOREMS FOR A ĆIRIĆ-REICH-RUS PAIR OF MAPPINGS IN METRIC SPACES WITH A DIRECTED GRAPH

Narongsuk Boonsri[1], Somchit Chotchaisthit[2], Satit Saejung[3 §]

[1,2]Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen, 40002, THAILAND

[2,3]The Centre of Excellence in Mathematics
Commission on Higher Education (CHE)
Sri Ayudthaya Road, Bangkok, 10400, THAILAND

Abstract: Let (X, d) be a complete metric space with a directed graph G. Inspired by the work of Bojor, we introduce a new contractive type condition between a pair of mappings $S, T : X \to X$ with respect to G. In particular, if $S = T$, we obtain the result of Bojor. Moreover, we also establish a result under an independent hypothesis.

An example showing that it is applicable in our setting but not in Bojor's is given.

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1. Introduction

The study of the existence of a fixed point of a given mapping has been considered widely since Banach established his famous fixed point theorem for

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[§]Correspondence author

contractions. The developments in the theory are the move from contractions to other mappings and from the complete metric space setting to other settings. Let (X, d) be a complete metric space. Recall that a mapping $T : X \to X$ is a *contraction* if there is $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$. Kannan [4] introduced the following mapping which is called Kannan's mapping: there exists $\beta \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq \beta(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$. Reich [7] generalized the preceding two concepts by introducing the following mapping: there are $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$ and $d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$ for all $x, y \in X$. On the other hand, Nieto and Rodriguez-Lopez [5, 6] proved a fixed point theorem for contractions in partially ordered complete metric spaces. Jachymski [3] extended the concept of a partial order by using a directed graph. Recently, Bojor[1] proved a fixed point theorem for Reich's mappings in the setting of a complete metric space with a directed graph. In this paper, we generalize Bojor's result. We also establish a supplement result to Bojor's and give an example which is applicable in our results but not in Bojor's.

2. Definitions and Known Lemmas

Let (X, d) be a metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subset E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$. The letter $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges, that is, $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

Definition 1. Let (X, d) be a metric space endowed with a directed graph G as described above. Let $S, T : X \to X$. We say that a pair of mappings (S, T) is a *G-Ćirić-Reich-Rus pair* if the following conditions hold:

(C1) if $(x, y) \in E(G)$ then $(Sx, Ty) \in E(G)$ and $(Tx, Sy) \in E(G)$;

(C2) there exist $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$ with the properties

$$d(Sx, Ty) \leq \alpha d(x, Sx) + \beta d(y, Ty) + \gamma d(x, y)$$

and

$$d(Tx, Sy) \leq \alpha d(x, Tx) + \beta d(y, Sy) + \gamma d(x, y)$$

for all $(x, y) \in E(G)$. In this case, we also say that (S, T) is a G-Ćirić-Reich-Rus pair with parameters α, β, γ.

Remark 2. (1) (S, T) is a G-Ćirić-Reich-Rus pair if and only if (T, S) is a G-Ćirić-Reich-Rus pair.
(2) If $S = T$, then a G-Ćirić-Reich-Rus pair (S, T) is just the one in [1, Definition 6].

For a mapping $T : X \to X$, we write $\mathrm{Fix}(T) = \{x \in X : x = Tx\}$.

Proprosition 3. *Let (S, T) be a G-Ćirić-Reich-Rus pair with parameters α, β, γ. Then*

(1) $d(Sx, Tx) \leq \alpha d(x, Sx) + \beta d(x, Tx)$ *for all* $x \in X$.

(2) $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

Proof. (1) Let $x \in X$. Since $(x, x) \in E(G)$, we have

$$\begin{aligned} d(Sx, Tx) &\leq \alpha d(x, Sx) + \beta d(x, Tx) + \gamma d(x, x) \\ &= \alpha d(x, Sx) + \beta d(x, Tx). \end{aligned}$$

(2) If $\mathrm{Fix}(T) = \varnothing = \mathrm{Fix}(S)$, then we are done. Now, we assume that $\mathrm{Fix}(T) \neq \varnothing$. Let $x \in \mathrm{Fix}(T)$. Then $x = Tx$ and by (1) we have

$$d(Sx, x) = d(Sx, Tx) \leq \alpha d(x, Sx) + \beta d(x, Tx) = \alpha d(x, Sx).$$

Thus $d(x, Sx) = 0$, and hence $x \in \mathrm{Fix}(S)$. This means that $\mathrm{Fix}(T) \subset \mathrm{Fix}(S)$. Similarly, if $\mathrm{Fix}(S) \neq \varnothing$, then $\mathrm{Fix}(S) \subset \mathrm{Fix}(T)$. Hence $\mathrm{Fix}(T) = \mathrm{Fix}(S)$. □

Definition 4. Let (X, d) be a metric space endowed with a directed graph G and $S, T : X \to X$. We say that the sequence $\{x_n\}$ is an (S, T)-*G-sequence starting from* $x \in X$ if

(1) $\{x_n\}$ is iteratively defined by

$$\begin{aligned} x_0 &= x \\ x_{2n+1} &= Sx_{2n} \\ x_{2n+2} &= Tx_{2n+1} \end{aligned}$$

for all $n \geq 0$ and

(2) $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$.

Remark 5. If (S,T) satisfies (C1) of Definition 1 and $(x,Sx)\in E(G)$, then we can construct an (S,T)-G-sequence $\{x_n\}$ starting from x.

Definition 6. Let (X,d) be a metric space endowed with a directed graph G and $S,T:X\to X$. Let $\{x_n\}$ be an (S,T)-G-sequence starting from $x\in X$. We say that (S,T) is *G-continuous with respect to* $\{x_n\}$ if $x_n\to p$ implies that there exists a strictly increasing sequences $\{n_k\}$ on $\mathbb{N}$ such that $Sx_{2n_k}\to Sp$ or $Tx_{2n_k+1}\to Tp$.

3. Common Fixed Point Theorems

Theorem 7. *Let (X,d) be a complete metric space endowed with a directed graph G and $S,T:X\to X$. Assume that (S,T) is a G-Ćirić-Reich-Rus pair such that there exists an element $x\in X$ with $(x,Sx)\in E(G)$. Let $\{x_n\}$ be an (S,T)-G-sequence starting from x.*

If one of the following conditions holds:

(C3a) (S,T) is G-continuous with respect to $\{x_n\}$;

(C4a) X satisfies the condition:

> *for any $\{z_n\}\subset X$ if $z_n\to p$ and $(z_n,z_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $(z_{n_k},p)\in E(G)$ for all $k\in\mathbb{N}$,*

then T and S have a common fixed point.

Proof. Put $K=\frac{\alpha+\gamma}{1-\beta}<1$. By (C2), we have

$$\begin{aligned}&d(x_{2n+1},x_{2n+2})\\&=d(Sx_{2n},Tx_{2n+1})\\&\le\alpha d(x_{2n},Sx_{2n})+\beta d(x_{2n+1},Tx_{2n+1})+\gamma d(x_{2n},x_{2n+1})\\&=\alpha d(x_{2n},x_{2n+1})+\beta d(x_{2n+1},x_{2n+2})+\gamma d(x_{2n},x_{2n+1}).\end{aligned}$$

Then

$$d(x_{2n+1},x_{2n+2})\le\frac{\alpha+\gamma}{1-\beta}d(x_{2n},x_{2n+1})=Kd(x_{2n},x_{2n+1}).$$

Similarly, we also have

$$d(x_{2n+2},x_{2n+3})$$

$$
\begin{aligned}
&= d(Tx_{2n+1}, Sx_{2n+2}) \\
&\leq \alpha d(x_{2n+1}, Tx_{2n+1}) + \beta d(x_{2n+2}, Sx_{2n+2}) + \gamma d(x_{2n+1}, x_{2n+2}) \\
&= \alpha d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n+2}, x_{2n+3}) + \gamma d(x_{2n+1}, x_{2n+2}).
\end{aligned}
$$

Then

$$d(x_{2n+2}, x_{2n+3}) \leq \frac{\alpha+\gamma}{1-\beta} d(x_{2n+1}, x_{2n+2}) = Kd(x_{2n+1}, x_{2n+2}).$$

Hence, by induction, we have

$$d(x_n, x_{n+1}) \leq K^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}.$$

Then

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} K^n d(x_0, x_1) < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, we have $x_n \to p$ for some $p \in X$.

Assume (C3a) holds. Then there exists a strictly increasing sequence $\{n_k\}$ on $\mathbb{N}$ such that $Sx_{2n_k} \to Sp$ or $Tx_{2n_k+1} \to Tp$. If $Sx_{2n_k} \to Sp$, then

$$d(p, Sp) = \lim_{k\to\infty} d(x_{2n_k}, Sx_{2n_k}) = \lim_{k\to\infty} d(x_{2n_k}, x_{2n_k+1}) = 0.$$

Then $Sp = p$. It follows from Proposition 3 that p is a common fixed point of S and T. If $Tx_{2n_k+1} \to Tp$, then we also obtain the same conclusion.

Assume that (C4a) holds. Then there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $(x_{n_k}, p) \in E(G)$ for all $k \in \mathbb{N}$. We may assume that there exists a further subsequence of $\{x_{n_k}\}$, still denoted by $\{x_{n_k}\}$, such that n_k is even and $(x_{n_k}, p) \in E(G)$ for all $k \in \mathbb{N}$. (If n_k is odd for all $k \in \mathbb{N}$, then the proof can be proceeded similarly.) This implies that $x_{n_k+1} = Sx_{n_k}$ and

$$
\begin{aligned}
d(p, Tp) &= \lim_{k\to\infty} d(x_{n_k+1}, Tp) \\
&= \lim_{k\to\infty} d(Sx_{n_k}, Tp) \\
&\leq \alpha \lim_{k\to\infty} d(x_{n_k}, Sx_{n_k}) + \beta d(p, Tp) + \gamma \lim_{k\to\infty} d(x_{n_k}, p) \\
&= \alpha \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) + \beta d(p, Tp) + \gamma \lim_{k\to\infty} d(x_{n_k}, p) \\
&= \beta d(p, Tp).
\end{aligned}
$$

Then $d(p, Tp) = 0$. Hence, p is a fixed point of T. It follows from Proposition 3 that p is a common fixed point of S and T. □

Similar to Theorem 7, we have the following result for a (T,S)-G-sequence starting from x where $(x,Tx) \in E(G)$.

Theorem 8. *Let (X,d) be a complete metric spaces endowed with a directed graph G and $S,T : X \to X$. Assume that (S,T) is a G-Ćirić-Reich-Rus pair such that there exists an element $x \in X$ with $(x,Tx) \in E(G)$. Let $\{x_n\}$ be a (T,S)-G-sequence starting from x.*

If one of the following conditions holds:

(C3b) (T,S) is G-continuous with respect to $\{x_n\}$;

(C4a) X satisfies the condition:

> *for any $\{z_n\} \subset X$ if $z_n \to p$ and $(z_n, z_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $(z_{n_k}, p) \in E(G)$ for all $k \in \mathbb{N}$,*

then S and T have a common fixed point.

Letting $S = T$ gives a result of Bojor [1].

Corollary 9. *Let (X,d) be a complete metric spaces endowed with a directed graph G and $T : X \to X$ be a mapping such that there exists an element $x \in X$ with $(x,Tx) \in E(G)$. Assume that T satisfies the conditions:*

(C1a) if $(x,y) \in E(G)$ then $(Tx,Ty) \in E(G)$;

(C2a) there exist $\alpha, \beta, \gamma \in [0,1)$ such that $\alpha + \beta + \gamma < 1$ with the property

$$d(Tx,Ty) \le \alpha d(x,Tx) + \beta d(y,Ty) + \gamma d(x,y)$$

for all $(x,y) \in E(G)$.

If one of the following conditions holds:

(C3c) T is orbitally G-continuous (see [3, Definition 2.4]), that is, for all $x, y \in X$ and any strictly increasing sequence $\{n_k\}$ on $\mathbb{N}$, $T^{n_k}x \to y$ and $(T^{n_k}x, T^{n_k+1}x) \in E(G)$ for all $k \in \mathbb{N}$ imply $T(T^{n_k}x) \to Ty$;

(C4b) X satisfies the condition:

> *for any $\{z_n\} \subset X$ if $z_n \to p$ and $(z_n, z_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $(z_{n_k}, p) \in E(G)$ for all $k \in \mathbb{N}$,*

then T has a fixed point.

Remark 10. Bojor [1] proved the result above with the assumption (C4b) while our result is established under the independent assumption (C3c).

The following example is slightly modified from [1, Example 3]. We note that there is an example which is applicable in our result but not in Bojor's.

Example 11. Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Define the directed graph G by $V(G) = X$ and

$$E(G) = \{(x, y) \in (0, 1] \times (0, 1] : x \geq y\} \cup \{(0, 0), (0, 1)\}.$$

Set $Sx = Tx = \frac{x}{4}$ for all $x \in [0, 1]$. It is easy to see that (X, d) is a complete metric space and (S, T) is a G-Ćirić-Reich-Rus pair with $\alpha = 0, \beta = 0, \gamma = \frac{1}{4}$. Clearly (S, T) is G-continuous with respect to $\{x_n\}$ for all (S, T) sequences $\{x_n\}$ starting from any $x \in [0, 1]$. Note that X does not satisfies condition (C4b) because $(z, 0) \in E(G) \Leftrightarrow z = 0$. It is clear that 0 is a common fixed point of S and T.

The following result is obtained in the setting of $\tilde{G}$.

Theorem 12. *Let (X, d) be a complete metric spaces endowed with a directed graph G and $S, T : X \to X$ be mappings such that there exists an element $x \in X$ with $(x, Sx) \in E(\tilde{G})$. Assume that S and T satisfy the conditions:*

(C1b) if $(x, y) \in E(\tilde{G})$ then $(Sx, Ty) \in E(\tilde{G})$;

(C2b) $\exists \alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$ with the property

$$d(Sx, Ty) \leq \alpha d(x, Sx) + \beta d(y, Ty) + \gamma d(x, y)$$

for all $(x, y) \in E(\tilde{G})$.

Let $\{x_n\}$ be an (S, T)-$\tilde{G}$-sequence starting from x.

If one of the following conditions holds:

(C3d) (S, T) is $\tilde{G}$-continuous with respect to $\{x_n\}$;

(C4c) X satisfies the condition:

for any $\{z_n\} \subset X$ if $z_n \to p$ and $(z_n, z_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $(z_{n_k}, p) \in E(\tilde{G})$ for all $k \in \mathbb{N}$,

then T and S have a common fixed point.

Proof. Choose $K = \max\{\frac{\alpha+\gamma}{1-\beta}, \frac{\beta+\gamma}{1-\alpha}\} < 1$. Then by the hypothesis (C2b), we can show that

$$d(x_n, x_{n+1}) \leq K^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}.$$

Hence $\{x_n\}$ is a Cauchy sequence. The rest of the proof follows similarly as that of Theorem 7. □

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