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OSCILLATORY BEHAVIOR OF THIRD ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study the oscillatory behavior of the following neutral differential equation

$$[(a(t)([x(t)+p(t)x(\delta(t))])'')^{\alpha}]' + q(t)f(x(\tau(t))) = 0.$$

Sufficient conditions are obtained so that every every solution is either oscillatory or converges to zero. In particular. we extend the results obtained in [6] by not assuming $a(t)$ is non-decreasing. Examples are provided to illustrate the main results.

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1. Introduction

In this paper we are concerned with the oscillatory behavior of third order

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neutral differential equation of the form

$$[a(t)([x(t)+p(t)x(\delta(t))]'')^{\alpha}]' + q(t)f(x(\tau(t))) = 0, t \geq t_0 \tag{1.1}$$

subject to the conditions:

(C_1) $a(t), p(t), q(t), \tau(t), \delta(t)$ are real continuous functions for $t \geq t_0$;

(C_2) $0 \leq p(t) \leq p < 1$ and α is a ratio of odd positive integers ;

(C_3) $q(t)$ is nonnegative and $\tau(t) \leq t,\ \delta(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$;

(C_4) $a(t)$ is positive and $A(t) = \int_{t_0}^{t} a^{-\frac{1}{\alpha}}(s)ds \to \infty$, as $t \to \infty$;

(C_5) $f : R \to R$ is continuous with $uf(u) > 0$ and $\frac{f(u)}{u^{\alpha}} \geqslant M > 0$ for $u \neq 0$.

By a solution of equation (1.1), we mean a function $x(t)$ in $C^2([T_x, \infty))$ for which $a(t)(z''(t))^{\alpha}$ is in $C'([T_x, \infty))$ and equation (1.1) is satisfied on some interval $[T_x, \infty)$ where $T_x \geqslant t_0$, and $z(t) = x(t) + p(t)x(\delta(t))$. We consider only those solutions $x(t)$ of equation (1.1) which satisfy sup $\{|x(t)| : t \geqslant T_x\} \geq 0$ for all $t \geqslant T_x$. We assume that equation (1.1) possesses such a solution , see [4]. As a customary, a solution of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory.

In recent years, there has been great interest in studying the oscillatory behavior of third order differential equations; see for example [1-3, 5-8], and the references cited therein. In [6], the authors obtained sufficient conditions for the oscillation of solutions to equation (1.1), under the assumption that $a(t)$ is nondecreasing. Here we establish similar results without any such condition on $a(t)$. We follow the same strategy as in [6], but with new estimates in Lemmas 2.3 and 2.4.

All functional inequalities are assumed to hold eventually; that is, for all t large enough. Note that if $x(t)$ is a solution so is $-x(t)$; so our proofs are done only for positive solutions. In Section 2, we present oscillation results for equation (1.1) and in Section 3, we provide some examples to illustrate the main results.

2. Oscillation Results

For a solution $x(t)$ of equation (1.1), we define the corresponding function

$$z(t) = x(t) + p(t)x(\delta(t)). \tag{2.1}$$

To obtain sufficient conditions for the oscillation of solution of equation (1.1), we need the following lemmas.

Lemma 2.1. *([1,Lemma 1]). Let $x(t)$ be a positive solution of equation (1.1) for $t \geq t_0$. Then there are only two possible cases for $t \geq t_0$:*

$$\begin{array}{ll} (i) & z(t) > 0, z'(t) > 0, z''(t) > 0, (a(t)(z''(t))^{\alpha})' < 0; \\ (ii) & z(t) > 0, z'(t) < 0, z'' > 0, (a(t)(z''(t))^{\alpha})' < 0. \end{array}$$

Lemma 2.2. *([1,Lemma 2]). Let $x(t)$ be a positive solution of equation (1.1) and let the corresponding function $z(t)$ satify case (ii) of Lemma 2.1. If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty} q(s)ds\right]^{\frac{1}{\alpha}} dudv = \infty, \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Lemma 2.3. *([3,Lemma 2.3]) Assume that $u(t) > 0, u'(t) \geq 0$ and $(a(t)(u'(t))^{\alpha})' \leqslant 0$ on $[t_0, \infty)$. Then for each $l \in (0,1)$, there exists a $T_l \geq t_0$ such that*

$$\frac{u(\tau(t))}{A(\tau(t))} \geq l\frac{u(t)}{A(t)} \quad for \quad t \geq T_l.$$

Lemma 2.4. *([3,Lemma 2.4]) Assume that $z(t) > 0, z'(t) > 0$, $z''(t) > 0, (a(t)(z''(t))^{\alpha})' \leqslant 0$ on $[T_l, \infty)$, then*

$$\frac{z(t)}{z'(t)} \geqslant \frac{a^{\frac{1}{\alpha}}(t)A(t)}{2} \quad for\ t \geqslant T_l.$$

For simplicity of notation, we introduce

$$Q(t) = Ml^{\alpha}(1-p)^{\alpha}q(t)a(\tau(t))\left(\frac{A(\tau(t))}{A(t)}\right)^{\alpha}\left(\frac{A(\tau(t))}{2}\right)^{\alpha}$$

with $l \in (0,1)$ *and* $t \geqslant T_l$.

Lemma 2.5. *Let $x(t)$ be a positive solution of equation* (1.1) *and the corresponding function $z(t)$ satisfy case* (i) *of Lemma 2.1. Then for $l \in (0,1)$, there exists a $T_l \geq t_0$ and a positive function $w(t)$ defined for all $t \geq T_l$ such that for $t \geq T_l$*

$$\int_t^\infty Q(s)ds < \infty, \qquad \int_t^\infty \left(\frac{w^{\alpha+1}(s)}{a(s)}\right)^{\frac{1}{\alpha}} ds < \infty, \tag{2.3}$$

and

$$w(t) \geq \int_t^\infty Q(s)ds + \alpha \int_t^\infty \left(\frac{w^{\alpha+1}(s)}{a(s)}\right)^{\frac{1}{\alpha}} ds. \tag{2.4}$$

Proof. Assume that $x(t)$ is a positive solution of equation (1.1) and the corresponding function $z(t)$ satisfies case (i) of Lemma 2.1. From (2.1), we have

$$x(t) = z(t) - p(t)x(\delta(t)) > z(t) - p(t)z(\delta(t)) \geqslant (1-p)z(t).$$

Using the last inequality in (1.1), we obtain

$$(a(t)(z''(t))^\alpha)' \leqslant (1-p)^\alpha q(t) z^\alpha(\tau(t)) \leqslant 0. \tag{2.5}$$

Define

$$w(t) = a(t)\left(\frac{z''(t)}{z'(t)}\right)^\alpha \tag{2.6}$$

then $w(t) > 0$, and satisfies

$$w'(t) \leqslant -Mq(t)(1-p)^\alpha \frac{z^\alpha(\tau(t))}{(z'(t))^\alpha} - \frac{\alpha}{a^{\frac{1}{\alpha}}(t)} w^{1+\frac{1}{\alpha}}(t). \tag{2.7}$$

From Lemma 2.3 with $u(t) = z'(t)$, we have

$$\frac{1}{z'(t)} \geqslant l \frac{A(\tau(t))}{A(t)} \frac{1}{z'(\tau(t))}, \quad t \geqslant T_l,$$

where l is the same as in $Q(t)$. Now (2.7) becomes

$$w'(t) \leqslant -Ml^\alpha q(t)(1-p)^\alpha \left(\frac{A(\tau(t))}{A(t)}\right)^\alpha \left(\frac{z(\tau(t))}{z'(\tau(t))}\right)^\alpha - \frac{\alpha}{a^{\frac{1}{\alpha}}(t)} w^{1+\frac{1}{\alpha}}(t).$$

From Lemma 2.4, we have $z(t) \geqslant \frac{a^{\frac{1}{\alpha}}(t)A(t)}{2} z'(t)$, so that

$$w'(t) + Q(t) + \frac{\alpha}{a^{\frac{1}{\alpha}}(t)} w^{1+\frac{1}{\alpha}}(t) \leqslant 0, \qquad t \geqslant T_l. \tag{2.8}$$

Since $Q(t) > 0$ and $w(t) > 0$ for $t \geqslant T_l$, it follows that $w'(t) \leqslant 0$ and $-w'(t) \geqslant \alpha \frac{w^{1+\frac{1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}$. Thus

$$\left(\frac{1}{w^{\frac{1}{\alpha}}(t)}\right)' > \frac{1}{a^{\frac{1}{\alpha}}(t)}.$$

Integrating the last inequality from T_l to t, and using the fact that $w^{-\frac{1}{\alpha}}(T_l) > 0$, we obtain

$$w(t) < \frac{1}{\left(\int\limits_{T_l}^{t} a^{-\frac{1}{\alpha}}(s)ds\right)^{\alpha}}$$

which in view of (C_4) implies that $\lim\limits_{t\to\infty} w(t) = 0$. Integrating the inequality (2.8) from t to T, $T > t > T_l$, we obtain

$$w(T) - w(t) + \int\limits_t^T Q(s)ds + \alpha \int\limits_t^T \left(\frac{w^{\alpha+1}(s)}{a(s)}\right)^{\frac{1}{\alpha}} ds \leqslant 0. \tag{2.9}$$

We now claim that $\int\limits_t^{\infty} Q(s)ds < \infty$. Otherwise, it follows from (2.9) that

$$w(T) \leqslant w(t) - \int\limits_t^T Q(s)ds \to -\infty \; as \; T \to \infty,$$

which is a contradiction. Hence the claim is proved. Similarly we can show that $\int\limits_t^{\infty} \left(\frac{w^{\alpha+1}(s)}{a(s)}\right)^{\frac{1}{\alpha}} ds < \infty$. Now letting $T \to \infty$ in (2.9) and using $w(T) \to 0$ as $T \to \infty$ we obtain (2.4). This completes the proof. □

Next we present oscillation results for equation (1.1).

Theorem 2.6. *Assume condition* (2.2) *holds. If*

$$\int\limits_{t_0}^{\infty} Q(t)dt = \infty,$$

then any solution of equation (1.1) *is either oscillatory or* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \geqslant t_0$. Then $z(t)$ satisfies case (i) or (ii) of Lemma 2.1.

Assume that $z(t)$ satisfies Lemma 2.1 (ii). By condition (2.2) and Lemma 2.2 we see that $\lim_{t\to\infty} x(t) = 0$.

Next we assume that $z(t)$ satisfies Lemma 2.1 (i). From the Lemma 2.5, we obtain

$$\int_{t_0}^{\infty} Q(t)dt < \infty,$$

which is a contradiction. This completes the proof. □

Theorem 2.7. *Assume condition* (2.2) *holds. If*

$$\liminf_{t\to\infty} \frac{1}{A_0(t)} \int_t^{\infty} \left(\frac{A_0^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds > \frac{1}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}, \tag{2.10}$$

where $A_0(t) = \int_t^{\infty} Q(t)dt$, *then any solution of equation* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality , we may assume that $x(t) > 0$ for all $t \geqslant t_0$. In view of Lemma 2.1 ,$z(t)$ satisfies Lemma 2.1 (i) or (ii).

First assume that $z(t)$ satisfies Lemma 2.1 (i). Define $w(t)$ by (2.6) and then from Lemma 2.5 that (2.4) holds. By (2.10), there exists a constant $\lambda > (\alpha+1)^{-\frac{\alpha+1}{\alpha}}$, so that

$$\liminf_{t\to\infty} \frac{1}{A_0(t)} \int_t^{\infty} \left(\frac{A_0^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds > \lambda. \tag{2.11}$$

From (2.4), we have

$$\frac{w(t)}{A_0(t)} \geqslant 1 + \frac{\alpha}{A_0(t)} \int_t^{\infty} \left(\frac{w^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds$$

$$= 1 + \frac{\alpha}{A_0(t)} \int_t^\infty \left(\frac{A_0^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} \left(\frac{w(s)}{A_0(s)} \right)^{\frac{\alpha+a}{\alpha}} ds, t \geq t_1. \quad (2.12)$$

Let $\mu = \inf_{t \geq t_1} \frac{w(t)}{A_0(t)}$, then $\mu \geq 1$ and from (2.11) and (2.12) we have

$$\mu \geqslant 1 + \alpha \mu^{\frac{(\alpha+1)}{\alpha}} \lambda \geq 1 + \alpha \left(\frac{\mu}{\alpha+1} \right)^{\frac{(\alpha+1)}{\alpha}}$$

that is,

$$\frac{\mu}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1} \left(\frac{\mu}{\alpha+1} \right)^{\frac{(\alpha+1)}{\alpha}},$$

which is a contradiction . This completes the proof. □

Next define

$$B(t) = \int_t^\infty q(s) a(\tau(s)) \left(\frac{A(\tau(s))}{A(s)} \right)^\alpha \left(\frac{A(\tau(s))}{2} \right)^\alpha ds,$$

then by Theorem 2.7 we obtain the following corollary.

Corollary 2.8. *Assume that condition* (2.2) *holds. If*

$$\liminf_{t \to \infty} \frac{1}{B(t)} \int_t^\infty \left(\frac{B^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds > \frac{1}{(1-p)(\alpha+1)^{\frac{\alpha+1}{\alpha}}} \quad (2.13)$$

then any solution $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t \to \infty} x(t) = 0$.

Proof. We shall show that condition (2.13) implies (2.10). Note that

$$\frac{1}{A_0(t)} \int_t^\infty \left(\frac{A_0^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds = \frac{l(1-p)}{B(t)} \int_t^\infty \left(\frac{B^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds. \quad (2.14)$$

From (2.13), we have for $l \in (0, 1)$

$$\liminf_{t \to \infty} \frac{1}{B(t)} \int_t^\infty \left(\frac{B^{\alpha+1}(s)}{a(s)} \right)^{\frac{1}{\alpha}} ds > \frac{1}{l(1-p)(\alpha+1)^{\frac{\alpha+1}{\alpha}}}. \quad (2.15)$$

Combining (2.14) with (2.15), we obtain (2.10). Hence by Theorem 2.7, the result follows. □

3. Examples

In this section we provide two examples to illustrate the main theorems.

Example 3.1 Consider the third order differential equation

$$\left(\frac{1}{t^3}(z''(t))^3\right)' + \frac{c}{t^6}x^3\left(\frac{t}{3}\right) = 0, \quad t \geqslant 1, \tag{3.1}$$

where $z(t) = x(t) + \frac{1}{2}x\left(\frac{t}{2}\right)$ and $c > 0$. Here $\alpha = 3, a(t) = \frac{c}{t^3}, q(t) = \frac{c}{t^6}, p(t) = \frac{1}{2}, \tau(t) = \frac{t}{3}$ *and* $\delta(t) = \frac{t}{2}$.

Note that

$$\int_1^\infty \int_v^\infty \left[\frac{1}{a(u)} \int_u^\infty q(s)ds\right]^{\frac{1}{3}} dudv = \int_1^\infty \int_v^\infty \left(u^3 \int_u^\infty \frac{1}{s^6}ds\right)^{\frac{1}{3}} dudv = \infty.$$

Further $A(t) = \int_1^t sds = \frac{t^2-1}{2}$. It is easy to see that all conditions of Corollary 2.8 are satisfied and hence any nonoscillatory solution of equation (3.1) tends to zero as $t \to \infty$.

Example 3.2 Consider the differential equation

$$\left(\frac{1}{t}z''(t)\right)' + \frac{16}{3t^4}x\left(\frac{t}{3}\right) = 0, \quad t \geqslant 1, \tag{3.2}$$

where $z(t) = x(t) + \frac{1}{2}x\left(\frac{t}{2}\right)$. Here $\alpha = 1, a(t) = \frac{1}{t}, q(t) = \frac{16}{3t^4}, p(t) = \frac{1}{2}, \tau(t) = \frac{t}{3}, \delta(t) = \frac{t}{2}$ *and* $A(t) = \frac{t^2-1}{2}$. It is easy to see that all conditions of Theorem 2.7 are satisfied. Hence every nonoscillatory solution of equation (3.2) tends to zero as $t \to \infty$. In fact $x(t) = \frac{1}{t}$ is one such solution of equation (3.2), since it satisfies the equation.

We conclude this paper with the following remark.

Remark 3.1 The results presented in [6] cannot be applicable to Examples 3.1 and 3.2 , since in this examples $a(t)$ is nonincreasing, so our results extend and complement to those obtained by Su and Xu [6].

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