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ON THE DIOPHANTINE EQUATION $(2^k - 1)^x + (2^k)^y = z^2$ WHERE k IS AN EVEN POSITIVE INTEGER

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Abstract: In this paper,we will apply the Catalan's conjecture in oder to give the solution in non-negative integer of the Diophantine equation $(2^k - 1)^x + (2^k)^y = z^2$ where k is an even positive integer.

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1. Introduction

In 2012 and 2013, Sroysang (see [3],[4])studied respectively the Diophantine equations $31^x + 32^y = z^2$ and $7^x + 8^y = z^2$ where x, y and z are non-negative integers. He has used the Catalan's conjecture [2] to solve such equations above.The former equation, he showed that there is no non-negative integer solution and the later one, there is an only one solution $(0, 1, 3)$.

Firstly, we have proved the equation $15^x + 16^y = z^2$ by applying the same method as Sroysang did and it turns out that $(1, 0, 4)$is an unique solution for this equation. It is naturally to be interested in the Diophantine equation of type $(2^k - 1)^x + (2^k)^y = z^2$ in general and eventually we are currently able to

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answer the solution of such an equation in case k is an even positive integer. More details are provided in the Theorem 1 and Theorem 2.

2. Preliminaries

Proposition 2.1. *[2](The Catalan's conjecture)* $(3,2,2,3)$ *is a unique solution* (a,b,x,y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x*and* y *are integers with* $\min\{a,b,x,y\} > 1$

Lemma 2.2. *The Diophantine equation* $1+(2^k)^y = z^2$*has no non-negative integer solution when k is an even positive integer.*

Proof. Let x, y and z be non-negative integers such that $1 + (2^k)^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. If $y = 1$, then we get $(z+2^{\frac{k}{2}})(z-2^{\frac{k}{2}}) = 1$. Thus $z + 2^{\frac{k}{2}} = 1$ and $z - 2^{\frac{k}{2}} = 1$. It follows that $z = 1$, which contradicts to the assumption. Now we are going to apply the Catalan's conjecture in case of $y \geq 2$. Thus we have $(2^k)^y \geq (2^k)^2$. This yields $z > 4$. Since $\min\{z, 2^k, 2, y\} > 1$, we obtain that $(3,2,2,3)$ is an only one solution of $z^2-(2^k)^y = 1$ by Proposition 2.1. Then we have $k = 1$ which leads to a contradiction. □

Lemma 2.3. *The only solution* (x, z) *in non-negative intergers to the Diophantine equation* $(2^k - 1)^x + 1 = z^2$ *is* $(1, 2^{\frac{k}{2}})$ *when k is an even positive integer.*

Proof. Let x and z be non-negative integers. If $x = 0$ then $z^2 = 2$ which is impossible. It is clear that $(1, 2^{\frac{k}{2}})$ is the solution of $(2^k - 1)^x + 1 = z^2$ as k is even. In case of $x \geq 2$. Then we have $(2^k - 1)^x \geq (2^k - 1)^2$. This implies that $z > 3$. Since $\min\{z, (2^k - 1), 2, x\} > 1$, so by the Proposition 2.1 we obtain $(3,2,2,3)$ is the only one solution of $z^2 - (2^k - 1)^x = 1$. Then we have $2^k = 3$ which is impossible. □

3. Result

Theorem 3.1. *Let k be an even integer at least* 4*. Then* $(1, 0, 2^{\frac{k}{2}})$ *is an only one solution in non-negative integers of the Diophantine equation* $(2^k - 1)^x + (2^k)^y = z^2$.

Proof. Let k be an even integer at least 4 and x, y, z be non-negative integers. We divide y into two cases as follows.

Case 1: $y = 0$. By Lemma 2.3, we have $(1, 2^{\frac{k}{2}})$ is the solution of the equation $(2^k - 1)^x + 1 = z^2$. It follows that $(1, 0, 2^{\frac{k}{2}})$ is the solution of the equation $(2^k - 1)^x + (2^k)^y = z^2$.

Case 2: $y \geq 1$.

Subcase 2.1 $x = 0$. It follows by Lemma 2.2 that the equation $1+(2^k)^y = z^2$ has no non-negative solution.

Subcase 2.2 $x \geq 1$. Since $2^k - 1$ is odd, $(2^k - 1)^x + (2^k)^y$ is odd. This implies that z^2 is odd and finally we get z is odd. Then $z^2 \equiv 1 \pmod 4$. Since $z^2 - (2^k)^y \equiv 1 \pmod 4$ it follows that $(2^k - 1)^x \equiv 1 \pmod 4$. But$(2^k - 1)^x \equiv 3^x \pmod 4$. Thus $3^x \equiv 1 \pmod 4$ and it is not hard to see that x is even.

Now, let us consider

$$(2^k)^y = z^2 - (2^k - 1)^{2n} \tag{1}$$

when n is a positive integer and then factor the equation (1) as follow.

$$(2^k)^y = (z + (2^k - 1)^n)(z - (2^k - 1)^n) \tag{2}$$

Let $z + (2^k - 1)^n) = 2^\alpha$ and $z - (2^k - 1)^n) = 2^\beta$ for some non-negetive integers α and β such that $\alpha > \beta$ and $\alpha + \beta = ky$.

Since $2(2^k - 1)^n = (z + (2^k - 1)^n) - (z - (2^k - 1)^n) = 2^\alpha - 2^\beta = 2^\beta(2^{\alpha-\beta} - 1)$ so $\beta = 1$ and $\alpha \geq 2$. Then we have $(2^k - 1)^n = 2^{\alpha-\beta} - 1$. If $\alpha = 2$ then $y = \frac{3}{k}$ which is impossible.

Next we will consider two cases which are $n \geq 2$ and $n = 1$ under the condition that $\alpha \geq 3$ so that we can reach a contradiction as follows. In case of $n \geq 2$, then we get the contradiction by applying the Proposition 2.1 to the equation $2^{\alpha-1} - (2^k - 1)^n = 1$. For the other case we have $2^k - 1 = 2^{\alpha-1} - 1$,which leads to get $k = \frac{2}{y-1}$, impossibly. □

Theorem 3.2. *The Diophantine equation* $3^x + 4^y = z^2$ *has exactly two non-negative integer solutions which are* $(1, 0, 2)$*and* $(2, 2, 5)$.

Proof. Let x, y and z be non-negative integers. We are going to consider two cases depending to y as follows.

Case1: $y = 0$. Then $3^x = z^2 - 1 = (z+1)(z-1)$. Let $z + 1 = 3^\alpha$ and $z - 1 = 3^\beta$ when α and β are non-negative integers such that $\alpha > \beta$ and $\alpha + \beta = x$. Thus $2 = 3^\alpha - 3^\beta = 3^\beta(3^{\alpha-\beta} - 1)$.It follows that $\beta = 0$ and $\alpha = 1$. Hence $x = 1$ and eventually we get $z = 2$.

Case2: $y \geq 1$.

Subcase2.1: $x = 0$. Then we have $(z+2^y)(z-2^y) = 1$ and we can conclude that $z + 2^y = 1$and $z - 2^y = 1$. Hence $z = 1$which is impossible.

Subcase2.2:$x \geq 1$. Since $z^2 = 3^x + 4^y \equiv 1 (\mathrm{mod}\ 2)$ so z is odd. This follows that $3^x = z^2 - 4^y \equiv 1 (\mathrm{mod}\ 4)$. Thus x is even. Then we have $2^{2y} = z^2 - 3^{2n} = (z - 3^n)(z + 3^n)$ when n is a positive integer. This concludes that $z + 3^n = 2^\lambda$ and $z - 3^n = 2^\gamma$ for some non-negative integers λ, γ with the conditions that $\lambda > \gamma$ and $\lambda + \gamma = 2y$. Since $2(3^n) = (z + 3^n) - (z - 3^n) = 2^\gamma(2^{\lambda-\gamma} - 1)$ so $\gamma = 1$ and $\lambda \geq 2$. Then we have $3^n = 2^{\lambda-1} - 1$. If $\lambda = 2$ then $n = 0$ which is impossible.

Next we will consider two cases which are $n \geq 2$ and $n = 1$ under the condition that $\lambda \geq 3$. In case of $n \geq 2$ we get the contradiction by applying Proposition 2.1 to the equation $2^{\lambda-1} - 3^n = 1$. In the other case, we can finally obtain the solution $(2, 2, 5)$ under the condition that $\lambda + \gamma = 2y$. □

4. Remark

In the work of Chotchaisthit [1] in 2013, he has answered the solutions of the Diophantine equation $p^x + (p+1)^y = z^2$ where x, y and z are non-negative intergers and p is a Mersenne prime. However, he hasn't given the solutions when p is not Mersenne prime and now by elementary method it is still an open problem. From Theorem 1, we have answered some of such a problem previously. In Theorem 2, we therefore give two solutions in case of $p = 3$.

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