International Journal of Pure and Applied Mathematics
Volume 94 No. 2 2014, 233-239
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v94i2.10

ijpam.eu

ON COMPLETE ARCS OF THE RIGHT NEARFIELD PLANE OF ORDER 9

Anna Sharafutdinova
Department of Mathematics and Physics
Chelyabinsk State Pedagogical University
Lenin Avenue 69, Chelyabinsk, 454080, RUSSIA

Abstract: We describe a method to find complete arcs in the right nearfield plane of order 9 based on the duality of right and left nearfield planes. By applying this method to the right nearfield plane of order 9, we were able to achieve a complete listing of all its complete k-arcs, up to isomorphism.

AMS Subject Classification: 05B25, 51E15
Key Words: automorphism group, isomorphism, k-arc, k-lateral, the left nearfield plane, the right nearfield plane

1. Introduction

An important issue which is part of the finite projective plane research is the study of k-arcs. It is well known that there are four finite projective planes of order 9: the Desarguesian plane, the Hughes plane, left nearfield and right nearfield planes as it was shown in [1].

This paper is devoted to construction of a complete k-arc list, up to isomorphism, in the case of the right nearfield plane of order 9. The hardness of the problem lies on the unknown collineation group for such plane.

Received: April 1, 2014

Next sections are organized as follows. In Section 2 usual definitions are stated. In Section 3 the description of our proposed method to address this key question and main result are presented.

2. Mathematical Preliminaries

We will avoid a thorough treatment for the projective plane theory. Instead, we will consider only those aspects relevant to our question, it is suggested [2] for further references.

Definition 1. Let $(P; L; I)$ be a triple, where P and L are disjoint sets, their elements are called points and lines respectively, and I is an incidence relation from P to L. Such triple $(P; L; I)$ is said to be a projective plane as long as obey the following conditions:
A1. Given any two points, there is a unique line incident with both of them.
A2. Given any two distinct lines, there is a unique point incident with both of them.
A3. There are four points such that, there is no line incident with more than two points of them.

Definition 2. A projective plane is called to be a finite projective plane if and only if it has a finite set of points (lines).

Definition 3. Let T be a finite projective plane, we said that T has order n, for $n > 1$, whether obeys the following additional axiom:

A4. There is a line incident with $n + 1$ points.

This latter definition has the following straightforward consequences:
(1) there are $n + 1$ points on every line;
(2) there are $n + 1$ lines through every point;
(3) there are $n^2 + n + 1$ points in the plane;
(4) there are $n^2 + n + 1$ lines in the plane.

As we already said, studying arcs within a finite projective plane is a key research question. We provide further essential definitions [3].

Definition 4. An arc in a finite projective plane is a set of points such that no three points are collinear, *i.e.* they are no concurrently on a same line.

Definition 5. A k-arc is an arc containing k points of the plane.

Definition 6. A k-arc is called a complete arc, whether it cannot be extended to a larger arc.

Definition 7. A complete k-arc is called an oval, if there are no $(k+1)$-arcs in this plane.

Let us to recall some usual remarks. Given an oval in a finite projective plane of order n, then it has $n+1$ points when n is odd, otherwise it has $n+2$ points. The latter case is called a hyperoval [4]. On the other hand, for any given complete arc, it follows that every point of the plane lies on at least a secant line to the arc. This ensured that the complete arcs are concerned with the notion "a blocking set" [5].

3. Method and Results

As we point out before, a complete list of arcs in the right nearfield plane of order 9 remains an open problem, mainly due to the unknown collineation group of it. We suggest a different approach to its solution, thereby a method independent of it. We claim to achieve a favorable method as follows.

Our method had been staged into 2 steps. Firstly, we list all basic k-laterals (a set of k lines, where there are no three concurrent lines), up to isomorphism, within the left nearfield plane of order 9 for $k = 4, 5, 6, 7, 8, 9$, and 10. Secondly, due to the duality of right and left nearfield planes it follows that each k-lateral is converted to a k-arc of the right nearfield plane.

Furthermore, a k-lateral searching algorithm had been briefly considered. It can be noticed that basic k-laterals are bound up with $(k-1)$-laterals within the left nearfield plane of order 9. So, a k-lateral can be obtained by adding an admissible line (a line with none of intersection points of lines of the $(k-1)$-lateral) to the basic $(k-1)$-lateral. Hence, we proceeded to find a set of admissible lines for each basic $(k-1)$-lateral by GAP [6], then each set is break into equivalence classes (orbits) under the automorphism group of their respective $(k-1)$-lateral. We choose a representative for each of these classes associated to a given $(k-1)$-lateral, then we add to it these representatives and compare all k-laterals yielded by this process. We observed that some k-laterals are isomorphic under the collineation group of the left nearfield plane, that is, two k-laterals are isomorphic if and only if there is a collineation such that they are transformed the one into the other. Thus, we follow by choosing a basic k-lateral and excluding from the list all those k-laterals isomorphic to it. Finally, such procedure yields a complete list of all basic k-laterals in the left nearfield plane.

It deserves to be noticed that there are complete k-laterals among basic ones when $k > 5$.

Definition 8. A k-lateral that cannot be extended to a $(k+1)$-lateral is said to be complete.

An interesting consequence falls from this definition: if the set of admissible lines of the k-lateral is empty, then this k-lateral is complete.

The searching procedure described above leads us to the following theorems.

Theorem 9. *Consider the left nearfield plane of order 9, then it holds each of the following facts: there are 19 types of basic 4-laterals, 75 types of basic 5-laterals, 220 types of basic 6-laterals, 193 types of basic 7-laterals, 53 types of basic 8-laterals, 3 types of basic 9-laterals, and 1 type of basic 10-laterals, up to isomorphism.*

Theorem 10. *There are 1 type of complete 6-laterals, 21 types of complete 7-laterals, 45 types of complete 8-laterals, 1 type of complete 9-laterals, and 1 type of complete 10-laterals, up to isomorphism, in the left nearfield plane of order 9.*

We can recall that any basic k-lateral of the left nearfield plane can be converted to a k-arc of the right nearfield plane, then we have the following theorem.

Theorem 11. *Consider the right nearfield plane of order 9, then it contains the following: 1 type of complete 6-arcs, 21 type of complete 7-arcs, 45 type of complete 8-arcs, 1 type of complete 9-arcs, and 1 type of complete 10-arcs, up to isomorphism.*

Searching results are listed in table 1, where $S_{\mathsf{i}}^{\mathsf{k}}$ is a complete k-arc with number i in lexicographic order, $|G_{\mathsf{i}}^{\mathsf{k}}|$ is an order of the automorphism group of the k-arc, $N_{\mathsf{i}}^{\mathsf{k}}$ is a number of k-arcs isomorphic to $S_{\mathsf{i}}^{\mathsf{k}}$.

4. Conclusion

We had described procedure for searching arcs in the right nearfield plane. This procedure is supported by the duality of right and left nearfield planes of order 9. By applying this method a full listing of the complete arcs, up to isomorphism, in the right nearfield plane of order 9 has been achieved, fulfilling our main goal. Furthermore, it is well known the connection between complete

Table 1: The list of complete arcs in the right nearfield plane of order 9

i	$S_{\mathsf{i}}^{\mathsf{k}}$	$\lvert G_{\mathsf{i}}^{\mathsf{k}}\rvert$	$N_{\mathsf{i}}^{\mathsf{k}}$
	$k = 6$		
49	0, 00, 2, 11, 23, 27	8	38880
	$k = 7$		
3	∞, 0, 00, 11, 23, 32, 65	2	155520
24	0, 00, 2, 01, 13, 24, 45	1	311040
26	0, 00, 2, 01, 13, 34, 38	2	155520
44	0, 00, 2, 01, 32, 45, 76	2	155520
48	0, 00, 2, 11, 13, 27, 42	2	155520
53	0, 00, 2, 11, 13, 34, 62	1	311040
57	0, 00, 2, 11, 13, 35, 58	1	311040
71	0, 00, 2, 11, 23, 35, 54	1	311040
78	0, 00, 2, 11, 23, 54, 67	2	155520
92	0, 00, 2, 13, 14, 37, 38	1	311040
101	0, 00, 2, 13, 14, 46, 82	2	155520
126	0, 00, 2, 13, 27, 68, 74	1	311040
127	0, 00, 2, 13, 27, 74, 82	2	155520
128	0, 00, 2, 13, 34, 45, 71	1	311040
150	0, 2, 10, 13, 34, 42, 56	1	311040
158	0, 2, 10, 13, 34, 47, 71	1	311040
159	0, 2, 10, 13, 34, 51, 57	2	155520
165	0, 2, 10, 13, 34, 57, 86	1	311040
167	0, 2, 10, 13, 34, 71, 86	1	311040
175	0, 2, 10, 21, 32, 44, 56	2	155520
179	0, 2, 10, 23, 31, 54, 78	2	155520
	$k = 8$		
1	∞, 0, 00, 11, 23, 32, 46, 75	1	311040
2	∞, 0, 00, 11, 23, 32, 46, 84	4	77760
3	∞, 0, 00, 11, 23, 46, 52, 67	1	311040
4	∞, 0, 00, 11, 23, 47, 64, 75	2	155520
5	∞, 0, 00, 11, 23, 47, 64, 82	3	103680
6	0, 00, 2, 01, 13, 14, 38, 45	2	155520
7	0, 00, 2, 01, 13, 14, 38, 82	2	155520
11	0, 00, 2, 01, 13, 17, 38, 55	2	155520
12	0, 00, 2, 01, 13, 17, 38, 66	2	155520
13	0, 00, 2, 01, 13, 17, 38, 82	1	311040
14	0, 00, 2, 01, 13, 17, 42, 45	2	155520

16	0, 00, 2, 01, 13, 34, 35, 77	1	311040
17	0, 00, 2, 01, 13, 34, 56, 62	1	311040
18	0, 00, 2, 01, 32, 33, 65, 67	8	38880
19	0, 00, 2, 01, 32, 33, 76, 87	2	155520
20	0, 00, 2, 01, 32, 35, 74, 76	2	155520
21	0, 00, 2, 01, 32, 38, 44, 45	4	77760
22	0, 00, 2, 01, 32, 38, 66, 67	16	19440
23	0, 00, 2, 01, 32, 38, 76, 87	4	77760
24	0, 00, 2, 01, 32, 44, 67, 83	4	77760
25	0, 00, 2, 11, 13, 24, 35, 72	1	311040
26	0, 00, 2, 11, 13, 27, 62, 68	2	155520
27	0, 00, 2, 11, 13, 34, 35, 87	1	311040
28	0, 00, 2, 11, 13, 34, 37, 45	1	311040
29	0, 00, 2, 11, 13, 35, 37, 72	1	311040
30	0, 00, 2, 11, 13, 35, 64, 68	2	155520
31	0, 00, 2, 11, 23, 24, 35, 72	1	311040
32	0, 00, 2, 11, 32, 37, 43, 45	2	155520
33	0, 00, 2, 11, 32, 43, 45, 76	1	311040
34	0, 00, 2, 11, 34, 35, 86, 87	4	77760
35	0, 00, 2, 13, 14, 38, 82, 87	1	311040
37	0, 00, 2, 13, 16, 37, 38, 51	1	311040
38	0, 00, 2, 13, 18, 34, 42, 86	1	311040
39	0, 2, 10, 11, 32, 33, 75, 76	6	51840
40	0, 2, 10, 11, 32, 34, 75, 78	2	155520
43	0, 2, 10, 11, 32, 36, 54, 55	8	38880
44	0, 2, 10, 11, 32, 54, 63, 88	2	155520
46	0, 2, 10, 13, 34, 35, 56, 57	3	103680
47	0, 2, 10, 13, 34, 35, 56, 72	2	155520
48	0, 2, 10, 13, 34, 35, 57, 78	2	155520
49	0, 2, 10, 13, 34, 45, 47, 68	2	155520
50	0, 2, 10, 13, 34, 45, 51, 62	2	155520
51	0, 2, 10, 21, 32, 44, 65, 88	6	51840
52	0, 2, 10, 23, 31, 42, 65, 84	4	77760
53	0, 2, 10, 23, 34, 45, 56, 88	1	311040
	$k = 9$		
3	0, 2, 10, 11, 32, 36, 43, 44, 78	4	77760
	$k = 10$		
1	0, 00, 2, 01, 13, 16, 24, 27, 35, 68	32	9720

arcs and blocking sets, so we are hopeful to find similar answers to related question, for example, by finding the blocking sets in the left nearfield plane of order 9.

5. Acknowledgements

This work is partially supported by the grant of the Russian Ministry of Education. The author is indebted to V. Vasilkov, M. Kipnis, R. Medina and C. Martinez, who all provided encouragement and helpful suggestions at various stages of this work.

References

[1] C.W.H. Lam, G. Kolesova, and L. Thiel, A computer search of finite projective planes of order 9, *Discrete Math.*, **92**, (1991), 187–195, **doi:** 10.1016/0012-365X(91)90280-F.

[2] F. Kárteszi, *Introduction to finite geometries*, Acad. Kiado, Budadpest (1976).

[3] R.H.F. Denniston, On arcs in projective planes of order 9, *Manuscripta math.*, **4**, (1971), 61–89, **doi:** 10.1007/BF01168905.

[4] B. Qvist, Some remarks concerning curves of the second degree in a finite plane, *Ann. Acad. Sci. Fenn.*, Ser. A, No. 134, (1952), 27 p.

[5] A.A. Bruen, and J.C. Fisher, Blocking sets and complete arcs, *Pacific J. Math.*, **53**, No. 1 (1974), 73–84, **doi:** 10.2140/pjm.1974.53.73.

[6] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.4; 2014. (http://www.gap-system.org)