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SOME NEW REMARKS ON $l^{=p}$ AND $l^{>p}$ FAMILIES

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Abstract: Wituła and Słota have distinguished two new subfamilies $l^{>p}$ and $l^{=p}$ of the classical spaces l^q, $q > 0$, of absolutely q-summable sequences (the first one is called the almost l^p, the second one – the exactly l^p), for every $p > 0$. In presented paper the new properties of families $l^{>p}$ and $l^{=p}$, connected with some infinite subfamilies of $l^{>p}$, are introduced.

We also prove that if for an increasing sequence $\{p_n\}$ of positive integers $\varphi(k)$ denotes the number of $\{p_n\}$ not exceeding k, i.e. φ denotes the counting function of sequence $\{p_n\}$, and $\varphi(k)$ possesses the Chebyshev estimation (like for the prime numbers) then $\{p_n\} \in l^{>1}$ and $\{p_{p_n}\} \in l^{=1}$.

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1. Introduction

Wituła and Słota in [9], [11] have distinguished two new subfamilies $l^{>p}$ and $l^{=p}$ of the classical spaces l^q, $q > 0$, of absolutely q-summable sequences (the first one is called the almost l^p, the second one – the exactly l^p):

$$l^{>p} := \Big(\bigcap_{q>p} l^q\Big)\backslash l^p,$$

and

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$$l^{=p} := l^p \setminus \Big(\bigcup_{0<q<p} l^q \Big),$$

for every $p > 0$. We note that discussion on any properties of families $l^{>p}$ and $l^{=p}$ can be always reduced to such discussion within families $l^{>1}$ and $l^{=1}$ after appropriate rescaling of the discussed sequences. We have (see [11] and [2], [6], [7]):

$$\left\{ k^{-1-(\log \log k)^{-\alpha}} \right\} \in l^{=1}$$

for every $\alpha > 0$. Let us also notice that $\{p_n\} \in l^{>1}$ and $\{p_{p_n}\} \in l^{=1}$ where $\{p_n\}$ is the increasing sequence of all prime numbers (see Theorem 8 and Remarks 10 and 11 given at the end of this paper). In papers [9], [11] a series of fundamental properties of families $l^{>p}$ and $l^{=p}$, for any $p > 0$, is actually presented, however it still remains far from considering this subject of research as exhausted.

Aim of this paper is to obtain the new properties of families $l^{>p}$ and $l^{=p}$ connected with distinguishing some infinite subfamilies of $l^{>p}$. During the investigation some interesting problems arose, part of which still remains unsolved. The internal nature of family $l^{>p}$ seems to be intriguing from the analytical side [11] and from the set-theory point of view – which can be also seen in presented paper, as well as from the topological side – especially in the topic of the Baire categories, which will be the subject of our research planned for the future.

In present paper the elements of sequences belonging to family $l^{\geq 1} := l^{>1} \cup l^{=1}$ will be denoted in the form of inverses of the positive numbers.

In the paper we will use the concept of the Sierpiński family of increasing sequences of positive integers. Family $\mathcal{S}$, with $\operatorname{card} \mathcal{S} = \mathfrak{c}$, of increasing sequences of positive integers will be called the Sierpiński family if for any $a, b \in \mathcal{S}$, $a = \{a_n\}_{n=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty}$, the sets of values of a and b are almost disjoint whenever $a \neq b$, which means that there exists $k \in \mathbb{N}$ such that $a_m \neq b_n$ for all $m, n \in \mathbb{N}$, $m, n \geq k$ (see [3]).

2. Main Results

Theorem 1. *There exists family $\mathcal{T} \subset l^{>1}$ of power of the continuum such that for any two different $\{t_n^{-1}\}, \{\tau_n^{-1}\} \in \mathcal{T}$ the sequence $\{(t_n + \tau_n)^{-1}\}$ belongs to $l^{=1}$.*

Proof. Let $\mathcal{S}$ be the Sierpiński family of increasing sequences of natural numbers. With each sequence $s = \{s_n\} \in \mathcal{S}$ it will be connected the sequence

$t(s) = \left\{t_n(s)^{-1}\right\}$ defined in the following way

$$t_n(s) = \begin{cases} k, & \text{if } n = s_k, \text{ for some } k \in \mathbb{N}, \\ n^{1+(\log\log n)^{-1}}, & \text{for other } n \in \mathbb{N}. \end{cases}$$

Let us notice that if $r, s \in \mathcal{S}$, $r \neq s$, then for the sufficiently large $n \in \mathbb{N}$ the inequalities hold

$$\frac{1}{2} n^{-1-(\log\log n)^{-1}} \leq (t_n(s) + t_n(r))^{-1} < n^{-1-(\log\log n)^{-1}}.$$

□

Remark 2. Theorem 1 can be associated with one more nontrivial fact. There exist sequences $\left\{t_n^{-1}\right\}, \left\{\tau_n^{-1}\right\} \in l^{>1}$, both nonincreasing and such that $\left\{\gamma_n^{-1}\right\} \in l^{=1}$ where $\gamma_n = \max\{t_n, \tau_n\}$, for every $n \in \mathbb{N}$ i.e. $\gamma_n^{-1} = \min\left\{t_n^{-1}, \tau_n^{-1}\right\}$ (see the first Example in [8], see also [12]). Then, from Theorem 2 in [8] it follows that

$$\liminf_{n\to\infty} \frac{n}{t_n} = \liminf_{n\to\infty} \frac{n}{\tau_n} = 0.$$

It is worth to notice in this moment one more property. Since $max\{t_n, \tau_n\} \leq t_n + \tau_n \leq 2\max\{t_n, \tau_n\}$, thus

$$\frac{1}{2\max\{t_n, \tau_n\}} \leq \frac{1}{t_n + \tau_n} \leq \frac{1}{\max\{t_n, \tau_n\}},$$

$$\frac{1}{2}\min\left\{t_n^{-1}, \tau_n^{-1}\right\} \leq (t_n + \tau_n)^{-1} \leq \min\left\{t_n^{-1}, \tau_n^{-1}\right\}.$$

It means that the series $\sum \min\left\{t_n^{-1}, \tau_n^{-1}\right\}$ and $\sum (t_n + \tau_n)^{-1}$ are either simultaneously convergent or simultaneously divergent.

Remark 3. Theorem 1 can be generalized with respect to any sequence $\left\{a_n^{-1}\right\} \in l^{>1}$ in the following way.

Theorem 4. *Let $\left\{a_n^{-1}\right\} \in l^{>1}$. Then there exists a family $\mathcal{T} \subset l^{>1}$ such that* card $\mathcal{T} = \mathfrak{c}$ *and for any two different sequences $\left\{t_n^{-1}\right\}, \left\{\tau_n^{-1}\right\} \in \mathcal{T}$ we have $\left\{(t_n + a_n)^{-1}\right\} \in l^{>1}$, $\left\{(\tau_n + a_n)^{-1}\right\} \in l^{>1}$ and $\left\{(t_n + \tau_n)^{-1}\right\} \in l^{=1}$.*

Proof. Let us fix the increasing sequence $\{k_n\}_{n=1}^{\infty}$ of natural numbers such that

$$\sum_{r=k_n}^{k_{n+1}-1} a_r^{-1} > 1 \quad \text{and} \quad \sum_{r=k_n}^{k_{n+1}-1} \left(r^{1+(\log\log r)}\right)^{-1+n^{-1}} > 1 \tag{1}$$

for every $n \in \mathbb{N}$. Let $\mathcal{S}$ be the Sierpiński family of increasing sequences of natural numbers. For each sequence $\mathbb{s} = \{s_n\}_{n=1}^{\infty} \in \mathcal{S}$ we define the following sequence $T = T(\mathbb{s}) = \{t_r\}_{r=1}^{\infty}$ of positive numbers

$$t_r = \begin{cases} a_r & \text{for every } r \in \bigcup_{n\in\mathbb{N}} \{\tau \in \mathbb{N} \colon k_{s_n} \le \tau < k_{1+s_n}\}, \\ r^{1+(\log\log r)^{-1}} & \text{for other } r \in \mathbb{N}. \end{cases} \tag{2}$$

Then we notice that

$$\sum_{r=k_{s_n}}^{k_{1+s_n}-1} (t_r + a_r)^{-1} = 2^{-1} \sum_{r=k_{s_n}}^{k_{1+s_n}-1} a_r^{-1} > 2^{-1},$$

for every $n \in \mathbb{N}$, which implies $\left\{(t_r + a_r)^{-1}\right\} \in l^{>1}$. Now let us choose $\mathbb{s} = \{s_n\}_{n=1}^{\infty}, \mathbb{w} = \{w_n\}_{n=1}^{\infty} \in \mathcal{S}$ and let $S = S(\mathbb{s}) = \{t_r\}$ and $W = W(\mathbb{w}) = \{\tau_r\}$ be the respective real sequences defined as in (2). If $\mathbb{s}$ and $\mathbb{w}$ are different then there exists $N = N(\mathbb{s}, \mathbb{w}) \in \mathbb{N}$ such that

$$t_n + \tau_n > n^{1+(\log\log n)^{-1}}$$

for every $n \in \mathbb{N}$, $n > N$, and set $\mathbb{N} \backslash (\mathbb{s} \cup \mathbb{w})$ is infinite which by condition (1) means that $\left\{(t_n + \tau_n)^{-1}\right\} \in l^{=1}$. □

The next theorem is also a trial of generalization of Theorem 1 connected with more subtle choice of family $\mathcal{T}$. We succeeded with this task only in case of the countable family $\mathcal{T}$, however it seems that the respective generalization on family $\mathcal{T}$ with $\operatorname{card} \mathcal{T} = \mathfrak{c}$ is only the matter of more advanced proving technique.

Theorem 5. *There exists countable (infinite) family $\mathcal{T} \subset l^{>1}$ possessing the following properties*

1. *for any two $\left\{t_n^{-1}\right\}, \left\{\tau_n^{-1}\right\} \in \mathcal{T}$ we have $\left\{(t_n + \tau_n)^{-1}\right\} \in l^{>1}$;*
2. *for any three $\left\{t_n^{-1}\right\}, \left\{\tau_n^{-1}\right\}, \left\{\lambda_n^{-1}\right\} \in \mathcal{T}$, if these sequences are pairwise different then $\left\{(t_n + \tau_n + \lambda_n)^{-1}\right\} \in l^{=1}$.*

Proof. Let the infinite sets N_{ij}, $i, j \in \mathbb{N}$, $i < j$, form the partition of the set of natural numbers and

$$\sum_{n\in N_{ij}} \frac{1}{n} = \infty$$

for any $i, j \in \mathbb{N}$, $i < j$. To each $i \in \mathbb{N}$ we assign the sequence $\{t_n(i)\}_{n=1}^{\infty}$ of real numbers, elements of which are defined in the following way

$$t_n(i) := \begin{cases} n, & \text{if } n \in \bigcup\limits_{j=i+1}^{\infty} N_{ij} \cup \bigcup\limits_{j=1}^{i-1} N_{ji}, \\ n^{1+(\log\log n)^{-1}}, & \text{for other natural numbers.} \end{cases}$$

Then for each pair $i, k \in \mathbb{N}$, $i < k$ the relations hold

$$t_n(i) + t_n(k) =$$
$$= \begin{cases} 2n, & \text{if } n \in N_{ik}, \\ n + n^{1+(\log\log n)^{-1}}, & \text{if } n \in \Big(\bigcup\limits_{j<l} N_{jl}\Big) \setminus N_{ik}, \text{ and} \\ & j = i \text{ or } j = k \text{ or } l = i \text{ or } l = k, \\ 2n^{1+(\log\log n)^{-1}}, & \text{for other natural numbers.} \end{cases}$$

Next, for any triple $i, k, u \in \mathbb{N}$, $i < k < u$, we get the following relation

$$t_n(i) + t_n(k) + t_n(u) =$$
$$= \begin{cases} 2n + n^{1+(\log\log n)^{-1}}, & \text{if } n \in N_{ik} \cup N_{iu} \cup N_{ku}, \\ n + 2n^{1+(\log\log n)^{-1}}, & \text{if } n \in \bigcup\limits_{j<l} N_{jl}, \\ & \text{where either } j \text{ or } l \text{ belongs} \\ & \text{to the set of indexes } \{i, k, u\} \\ & \text{and } n \notin (N_{ik} \cup N_{iu} \cup N_{ku}), \\ 3n^{1+(\log\log n)^{-1}}, & \text{for other positive integers,} \end{cases}$$

which easily implies that $\left\{(t_n(i) + t_n(k) + t_n(u))^{-1}\right\} \in l^{=1}$. □

Remark 6. Theorem 4 can be generalized to the following form: for every $r \in \mathbb{N}$ there exists an infinite countable family $\mathcal{T} \subset l^{>1}$ such that

1. for any $\left\{t_{k,n}^{-1}\right\}_{n=1}^{\infty} \in \mathcal{T}$, $k = 1, 2, \ldots, r$, we have
$$\left\{\Big(\sum_{k=1}^{r} t_{k,n}\Big)^{-1}\right\}_{n=1}^{\infty} \in l^{>1},$$
2. for any $\left\{t_{k,n}^{-1}\right\}_{n=1}^{\infty} \in \mathcal{T}$, $k = 1, 2, \ldots, r+1$, if these sequences are pairwise disjoint then $\left\{\Big(\sum_{k=1}^{r+1} t_{k,n}\Big)^{-1}\right\}_{n=1}^{\infty} \in l^{=1}$.

3. Supplementary Facts

Let us present two more theorems of existential nature concerning the elements of sets $l^{>1}$ and $l^{=1}$. Especially surprising for the Authors is Theorem 8 related to the increasing sequences of natural numbers, called by the Authors as the asymptotic prime numbers, the counting function of which satisfies the Chebyshev estimation for prime numbers.

Theorem 7. *Let $\{a_n\}$ be a sequence of positive reals converging to zero. If $\sum a_n = \infty$ then there exist two nonincreasing sequences $\{b_n\}$ and $\{d_n\}$ of positive reals satisfying the following conditions*

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = 0,$$

$$\sum a_n b_n = \infty, \qquad \sum a_n d_n < \infty,$$

$$\sum a_n (b_n)^u < \infty \qquad \text{and} \qquad \sum a_n (d_n)^v = \infty$$

for every $u > 1$ and $v < 1$.

Proof. Let us fix the increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$1 - 2^{-k} < \sum_{i=1+n_k}^{n_{k+1}} a_i < 1 + 2^{-k},$$

for every $k \in \mathbb{N}$. Next, it is sufficient to define $b_i = k^{-1}$ and $d_i = k^{-1-(\log\log k)^{-1}}$ for every $i \in \mathbb{N}$, $n_k < i \le n_{k+1}$ and $k \in \mathbb{N}$. □

The special case of Theorem 7, obtained by using Abel-Dini Theorem, is presented by Hildebrandt in [4].

Theorem 8. *Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers and let $\varphi(n)$ denote the number of elements of this sequence not exceeding n, for every $n \in \mathbb{N}$. If there exist $\alpha, \beta > 0$, such that for sufficiently large $n \in \mathbb{N}$ the inequalities hold*

$$\alpha \frac{n}{\log n} < \varphi(n) < \beta \frac{n}{\log n}, \tag{3}$$

then

$$\{p_n^{-1}\} \in l^{>1} \quad \text{and} \quad \{p_{p_n}^{-1}\} \in l^{=1}.$$

Proof. At first we show that for sufficiently large $n \in \mathbb{N}$ the inequalities hold

$$an \log n < p_n < bn \log n \tag{4}$$

for some $b > a > 0$.

Suppose that the inequalities (3) are satisfied for $n \geq m$. Then we receive

$$\alpha \frac{p_n}{\log p_n} < \varphi(p_n) < \beta \frac{p_n}{\log p_n}, \ \ n \geq m,$$

which results from the fact that $p_n \geq n$, $n \in \mathbb{N}$. But $\varphi(p_n) = n$, therefore

$$\alpha \frac{p_n}{\log p_n} < n < \beta \frac{p_n}{\log p_n}, \ \ n \geq m. \tag{5}$$

Taking logarithm of (5) and multiplying next both sides of the obtained inequality, respectively, by inequality (5) we get

$$\alpha \frac{p_n}{\log p_n}(\log \alpha + \log p_n - \log\log p_n) < n \log n < \\ < \beta \frac{p_n}{\log p_n}(\log \beta + \log p_n - \log\log p_n),$$

i.e.,

$$\alpha p_n - \alpha p_n \frac{\log\log p_n - \log \alpha}{\log p_n} < n \log n < \\ < \beta p_n - \beta p_n \frac{\log\log p_n - \log \beta}{\log p_n}. \tag{6}$$

Since

$$\frac{\log\log p_n - \log \alpha}{\log p_n} \xrightarrow[n\to\infty]{} 0$$

and

$$\log\log p_n \xrightarrow[n\to\infty]{} \infty,$$

thus from (6) it results that for sufficiently large $n \in \mathbb{N}$ the inequalities hold

$$\frac{\alpha}{2} p_n < n \log n < \beta p_n,$$

from which we obtain

$$\frac{1}{\beta} n \log n < p_n < \frac{2}{\alpha} n \log n.$$

In this way we proved (4) where, as it can be seen, one can take $a = \frac{1}{\beta}$ and $b = \frac{2}{\alpha}$. Inequality (4) implies that for sufficiently large $n \in \mathbb{N}$ the following inequality

$$\frac{1}{p_n} > \frac{1}{bn \log n}$$

is fulfilled. Hence, in view of divergence of series $\sum\limits_{n=1}^{\infty} \frac{1}{n \log n}$ we get $\sum\limits_{n=1}^{\infty} \frac{1}{p_n} = \infty$.

From (4) we obtain also the following estimation

$$p_{p_n} > ap_n \log p_n > an \log n \log(an \log n)$$

(for sufficiently large $n \in N$), from which, since $a \log(n) \xrightarrow[n\to\infty]{} \infty$, we get

$$p_{p_n} > an(\log n)^2,$$

that is

$$\frac{1}{an(\log n)^2} > \frac{1}{p_{p_n}}. \tag{7}$$

Hence, by convergence of series $\sum\limits_{n=1}^{\infty} \frac{1}{n(\log n)^2}$, the convergence of series $\sum\limits_{n=1}^{\infty} \frac{1}{p_{p_n}}$ results. From (4) we receive also the estimation (for sufficiently large $n \in \mathbb{N}$):

$$p_{p_n} < bp_n \log p_n < b^2 n \log n \log(bn \log n) < 2b^2 n(\log n)^2$$

which means that $\sum p_{p_n}^{-x} \geq \sum (2b^2 n \log^2 n)^{-x} = \infty$, for every $x < 1$. □

Corollary 9. *If the sequence $\{p_n\}_{n=1}^{\infty}$ is formed by the successive prime numbers then $\{p_n^{-1}\} \in l^{>1}$ and $\{p_{p_n}^{-1}\} \in l^{=1}$.*

Proof. The prime number theorem (see [1, Chapter 4]) implies that there exist constants $b > a > 0$ such that

$$a\frac{n}{\log n} < \pi(n) < b\frac{n}{\log n}$$

for $n \geq 2$, where $\pi(n)$ denotes the number of primes not exceeding n for every $n \geq 2$. □

Remark 10. If condition (3) is replaced by the following one

$$\alpha\frac{n}{\log \log n} < \varphi(n) < \beta\frac{n}{\log \log n} \tag{8}$$

for sufficiently large $n \in \mathbb{N}$, then similarly as in the proof of Theorem 8 it could be proven that $\left\{\left(p_n^{(k)}\right)^{-1}\right\}_{n=1}^{\infty} \in l^{>1}$ for every $k \in \mathbb{N} \cup \{0\}$, where

$$p_n^{(0)} := p_n, \quad p_n^{(k+1)} := p_{p_n}^{(k)}, \text{ for every } k \in \mathbb{N}.$$

Whereas if condition (3) is replaced by the new one

$$\alpha \frac{n}{(\log n)^{\omega}} < \varphi(n) < \beta \frac{n}{(\log n)^{\omega}}, \tag{9}$$

where $\omega \in (0,1)$, then in the same way as in the proof of Theorem 8 it could be proven that

$$\left\{\left(p_n^{(k)}\right)^{-1}\right\}_{n=1}^{\infty} \in l^{>1} \text{ and } \left\{\left(p_n^{(k_0)}\right)^{-1}\right\}_{n=1}^{\infty} \in l^{=1},$$

for every $k = 0, 1, \ldots, k_0 - 1$, where $k_0 := \min\{\tau \in \mathbb{N} : \tau\omega > 1\}$. Hence, for example, for $\omega = \frac{1}{2}$ we get that

$$\{p_n\} \in l^{>1}, \ \{p_{p_n}\} \in l^{>1} \text{ and } \{p_{p_{p_n}}\} \in l^{=1}.$$

All the above results can be generalized to the respective families of increasing sequences of positive integers. For example, if two increasing sequences $\{p_n\}$ and $\{q_n\}$ of positive integers possess the counting functions $\varphi(n)$ and $\psi(n)$, respectively, which satisfy condition (3) then $\{p_{q_n}\} \in l^{=1}$.

Next, if three increasing sequences $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ of positive integers satisfy condition (9) with the following powers ω: two sequences with $\omega = \frac{1}{3}$ and one sequence with $\omega = \frac{2}{3}$ (which one is exactly connected with the respective power is unimportant) then $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{p_{q_n}\}$, $\{q_{p_n}\}$, $\{p_{r_n}\}$, $\{r_{p_n}\}$, $\{q_{r_n}\}$, $\{r_{q_n}\}$ all belong to $l^{>1}$, whereas $\{p_{q_{r_n}}\}$, $\{q_{p_{r_n}}\}$, $\{p_{r_{q_n}}\}$, $\{r_{p_{q_n}}\}$, $\{q_{r_{p_n}}\}$, $\{r_{q_{p_n}}\}$ all belong to $l^{=1}$.

Remark 11. There exists one more attractive property of the sequence $\{p_n\}_{n=1}^{\infty}$ of positive integers possessing the counting function $\varphi(n)$ satisfying the following condition $\varphi(n) \sim \frac{n}{\log n}$ as $n \to \infty$, which is stranger than condition (3). Montgomery in [5] proved that number 1 is then the limit point of sequence $\{f(n)\}_{n=1}^{\infty}$, where

$$f(n) = \sum_{p_k < n} \frac{1}{n - p_k}$$

for every $n \in \mathbb{N}$. In this way he answered affirmatively the question posed by Erdös.

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