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PERFORMANCE MEASURE OF LAPLACE TRANSFORMS FOR PRICING PATH DEPENDENT OPTIONS

Chuma Raphael Nwozo[1 §], Sunday Emmanuel Fadugba[2]

[1]Department of Mathematics
University of Ibadan
Oyo State, NIGERIA

[2]Department of Mathematical Sciences
Ekiti State University
Ado Ekiti, NIGERIA

Abstract: This paper presents a performance measure of Laplace transforms for pricing path dependent options. We obtain a simple expression for the double transform by means of Fourier and Laplace transforms, (with respect to the logarithm of the strike and time to maturity) of the price of continuously monitored Asian options. The double transform is expressed in terms of Gamma functions only. The computation of the price requires a multivariate numerical inversion. Under jump-diffusion model, we show that the Laplace transforms of lookback options can be obtained through a recursion involving only analytical formulae for standard European call and put options. We also show that the numerical inversion can be performed with great accuracy and low computational cost.

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1. Introduction

As stock markets have become more sophisticated, so have their products. The

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§Correspondence author

simple buy or sell trades of the early markets have been replaced by more complex financial options and derivatives. These contracts can give investors various opportunities to tailor their deals to their investment needs.

One of the main concerns of financial derivatives is to obtain the exact values of options. For the simplest model in the case of constant coefficients, an exact pricing formula was derived by Black and Scholes- the well known Black-Scholes formula. However, in the general case of time and space dependent coefficients the exact pricing formulae are not yet known, and thus numerical solutions have been used.

Financial engineers have created various exotic products to meet the different market needs. These products are designed to meet a genuine hedging need in the market, i.e. they are "tailor-made" and to reflect a corporate treasurer's view on potential future movements in particular market variables.

A path dependent option is a contingent claim whose value depends on the sequence of prices of the underlying asset during the whole or part of the option's life rather than just the final price of the asset. Examples of path dependent options are Asian and lookback options. Asian options are path dependent options whose payoff functions depend on the average stock price over a specific period of time called the life of the option. The Asian Call option gives the holder of the option the right to buy an underlying security but under no obligation. The Asian put option gives the holder the right to sell. There are two types of Asian options with regards to the average computation: arithmetic and geometric. For the geometric Asian options there is a closed-form solution to the value of these options. However, this is not the case for the arithmetic type because the arithmetic average of a set of lognormal random variables is not lognormally distributed. Until now there has been no closed-form solution to the value of these types of option [21]. Asian options reduce the possibility of market manipulation near the expiration date and offer a better hedge to firms with a stream of positions. Because of this, they have lower volatility and hence rendering them cheaper relative to their European counterparts. The lookback option is defined as a financial derivative whose strike price corresponds to the minimum or maximum price recorded by the underlying asset during the option's life. Lookback call(put) gives the option the right to buy(sell) an asset at its lowest(highest) price during the life of the option. Obviously, the more flexible the option, the more expensive it will be.

In the recent years the complexity of numerical computation in financial theory and practice has increased greatly, putting more demands on computation speed and efficiency.

Numerical methods are needed for pricing options in cases where analytic solutions are either unavailable or not easily obtained. They are used for a variety of purpose in finance. These include the valuation of securities, the estimation of their sensitivities, risk analysis and stress testing of portfolios.

In this paper we consider the pricing of Asian options by computing a Laplace transform with respect to time. The Fourier transform with respect to the logarithm of the strike price is used as a technique to invert Laplace transform. Such double transform is related to the characteristic function of a normal random variable. The double transform can easily be expressed in terms of gamma functions. [1] show that the double transform is easily extend to the computation of the greeks, like delta and gamma. In order to numerically invert the double transform and obtain the option price, they use a multivariate version of the Fourier-Euler algorithm. They show that this numerical inversion is highly accurate with low volatility levels compare to Monte Carlo method.

Under the standard Black-Scholes framework, the arithmetic average of prices is a sum of correlated lognormal distributions. Since the distribution of this sum does not admit a simple analytical expression, several approaches have been proposed to price Asian and lookback options [8]. The Laplace transform approach was considered by H. Geman and M. Yor [6]. They used a Laplace transform to price Asian option. They also exploit the relationship between the Geometric Brownian motion and the Bessel process with a stochastic time change and the additivity property of the Bessel process. Moreover, the numerical inversion of the Laplace transform given in [6] cannot be performed at low volatility levels, due to limited computer precision. Instead, the numerical inversion of the double transform can be computed with accuracy at low volatility. The logarithmic moments was considered by G. Fusai and A. Tagliani [5].

Monte Carlo method for pricing some path dependent options was considered by C. R. Nwozo and S. E. Fadugba [10]. The competitive Monte Carlo methods for the pricing of Asian options was considered by B. Lapeyre and E. Temam [9].

G. Petrella and S. G. Kuo [12] used a laplace transform to price discretely monitored barrier and lookback options. L. C. G. Rogers and Z. Shi [13] transformed the problem of pricing Asian options into a means of solving a parabolic partial differential equations in two variables from the second order, but there is no analytical solution for this partial differential equation, and its numerical solution is not accurate. They also derive lower bound formula for Asian options by computing the expectation based on a zero-mean Gaussian variable.

The approach used in this paper is different from M. C. Fu et al [4] who investigated a double transform of the option price but with respect to time and strike price. They obtained a complicated expression in terms of non-standard functions, since their result is related to the Laplace transform of a lognormal variable, which does not admit an analytical expression. Moreover, their double transform proves hard to invert numerically.

For a survey of different Laplace inversion algorithms and their performances in derivatives pricing see ([2], [3], [8], [15], [14], [16], [17], [18], [19], [20]) just to mention few.

In this paper, we shall consider a performance measure of the Laplace transform for pricing path dependent options namely; Asian and lookback options.

2. The Double Transform for Pricing Asian Option

This section presents the double transform for pricing Asian option.

2.1. The Laplace Transform for the Asian Option

We begin with the assumption that the risk-neutral process for the underlying asset is given by a stochastic differential equation.

$$dS = rSdt + \sigma SdW_t \tag{1}$$

where W_t is a Brownian motion or wiener process, r is the interest rate, t is the time and σ is the volatility.

Under this condition, in order to price continuously monitored Asian option, we need the probability density function of the random variable S i.e.

$$A_t = \int_0^t \exp\left(\left(r - \frac{\sigma^2}{2}\right) + \sigma W_s\right) dS \tag{2}$$

The payoff of a fixed strike Asian option is given by

$$P_A = \max\left(\frac{S_0 A_t}{t} - K, 0\right) \tag{3}$$

The case of floating strike Asian options which is characterized by a payoff $\max\left(\frac{S_0 A_t}{t} - S_t, 0\right)$ can be dealt with by using the parity result in [7]. The presence of a continuous dividend yield q can be taken into account in order to replace r by $(r-q)$ and the spot price by $S_0 e^{-qt}$. If the interest rate or volatility is not constant, then the pricing of the Asian option becomes more difficult.

We obtain the price of the Asian option by computing the discounted expected value:

$$e^{-rt}E_0 \max\left(\frac{S_0 A_t}{t} - K, 0\right) = e^{-rt}\frac{S_0}{t}E_0 \max(A_t - J, 0) \tag{4}$$

where E_0 is the expected value under the risk-neutral probability measure and $J = (\frac{K}{S_0})t$. In order to compute this expectation, we first use the scaling property of the Brownian motion to express A_t as

$$A_t = \frac{4}{\sigma^2} D^{(v)}_{\frac{\sigma^2 t}{4}} \tag{5}$$

where

$$D_h^{(v)} \equiv \int_0^h e^{2(W_s + v_s)} ds \tag{6}$$

and $v = \frac{2r}{\sigma^2 - 1}$. Thus we obtain

$$\begin{aligned} E_0(A_t - J)^+ &= E_0 \max\left(\frac{4}{\sigma^2} D_h^{(v)} - J, 0\right) \\ &= \frac{4}{\sigma^2} E_0 \max\left(D_h^{(v)} - J, 0\right) \\ &= \frac{4}{\sigma^2} \int_R^\infty (x - J) f_D(x, h) dx \end{aligned} \tag{7}$$

where f_D is the density function of the random variable $D_h^{(v)}$; $J \equiv K\frac{4P}{\sigma^2}$; and $t \equiv \frac{4h}{\sigma^2}$. After a final change of variable, $w = \ln x$, we are interested in the function:

$$c(k, h) \equiv \frac{4}{\sigma^2} \int_k^\infty (e^w - e^k) f_{\ln D}(w, h) dw \tag{8}$$

where $k = \ln P$. Note that we have used the fact that the density law of the logarithm of a random variable is related to the density of the same random variable by the relationship:

$$f_{\ln D} = f_D(e^\omega, h)e^\omega, -\infty < \omega < \infty \tag{9}$$

We shall compute the analytical expression of the double transform $c(k, h)$ for Laplace and Fourier with respect to h and k respectively. Following [4], we multiply (8) by an exponentially decaying function $e^{-a_f k}$, $c(k, h)$ becomes square integrable in k over the negative axis. Therefore, we replace the function

$c(k,h)$ by $c(k,h;a_f)$, where $c(k,h;a_f) \equiv c(k,h)e^{-a_f k}$, $a_f > 0$, and we compute the double of $c(k,h;a_f)$:

$$L(F(c(k,h;a_f);k \to \gamma);h \to \lambda)) \equiv \int_0^\infty e^{-\lambda h} \int_{-\infty}^\infty e^{i\gamma k} c(k,h;a_f) dk dh \tag{10}$$

Theorem 1. *The double transform of $c(k,h;a_f)$, for $\lambda > 2\gamma(\gamma+\nu)$ gives*

$$\mathrm{L}(F(c(k,h;a_f);k \to \gamma);h \to \lambda) = C(\gamma + ia_f, \lambda)$$

where

$$C(\gamma,\lambda) = \frac{4\Gamma(i\gamma)\Gamma(\frac{\mu+\nu}{2}+1)\Gamma(\frac{\mu-\nu}{2}-1-i\gamma)}{\gamma^2\lambda 2^{1+i\gamma}\Gamma(\frac{\mu+\nu}{2}+2+i\gamma)\Gamma(\frac{\mu-\nu}{2})} \tag{11}$$

and $\Gamma(.)$ is the gamma function of complex argument and $\mu^2 = 2\lambda + \nu^2$. Also, we can obtain the delta and gamma of the Asian option respectively. After some algebra we have that:

$$\Delta(S_0,K,t,r,\sigma) \equiv e^{-rt}\frac{\partial}{\partial S_0}(E_0 \max(A_t - J))$$

$$= \frac{e^{-rt}}{t}\left(c(k,h) - \frac{\partial c(k,h)}{\partial k}\right)|_{k=\ln(\frac{K\sigma^2 t}{4S_0}),h=\frac{\sigma^2 t}{4}} \tag{12}$$

Equation (12) is called the delta of the Asian option. We also obtain the gamma of the Asian option from the gamma function

$$\Gamma(S_0,K,t,r,\sigma) \equiv e^{-rt}\frac{\partial^2}{\partial S_0^2}(E_0 \max(A_t - J))$$

This reduces to

$$\Gamma(S_0,K,t,r,\sigma) = \frac{e^{-rt}}{S_0 t}\left(\frac{\partial c(k,h)}{\partial k} - \frac{\partial^2 c(k,h)}{\partial k^2}\right)|_{k=\ln(\frac{K\sigma^2 t}{4S_0}),h=\frac{\sigma^2 t}{4}} \tag{13}$$

Equation (13) is the gamma of the Asian option.

The delta and gamma of the Asian option can also be recovered by numerically inverting their double transforms:

$$A(\gamma + ia_f, \lambda) = \int_0^\infty e^{-\lambda h} \int_{-\infty}^\infty e^{-(a_j k + rt)}\left(c(k,h) - \frac{\partial c(k,h)}{\partial k}\right) dk dh \tag{14}$$

$$B(\gamma + ia_f, \lambda) = \int_0^\infty e^{-\lambda h} \int_{-\infty}^\infty e^{-(a_j k + rt)}\left(\frac{\partial c(k,h)}{\partial k} - \frac{\partial^2 c(k,h)}{\partial k^2}\right) dk dh \tag{15}$$

where $A(\gamma,\lambda)$ and $B(\gamma,\lambda)$ are a simple rescaling of the function $C(\gamma,\lambda)$ given below respectively:

$$A(\gamma,\lambda) = (1+i\gamma)C(\gamma,\lambda),$$
$$B(\gamma,\lambda) = i\gamma(1+i\gamma)C(\gamma,\lambda)$$

2.2. Numerical Inversion for Asian Option

To obtain the function $c(k,h)$ by the double numerical inversion, we begin with the price of the Asian option given by

$$e^{-rt}E_0 \max\left(\frac{S_0 A_t}{t} - K, 0\right) = e^{-rt}\frac{S_0}{t}e^{a_f k}c(k,h;a_f)|_{k=\ln(\frac{K\sigma^2 t}{4}),h=\frac{\sigma^2 t}{4}} \tag{16}$$

The numerical Inversion of the double transform in (11) can be performed by resorting to the multivariate version of the Fourier-Euler algorithm given by

$$\bar{f}(t_1,t_2) = \frac{e^{\frac{A_1}{2l_1}}}{2l_1 t_1}\sum_{j=-\infty}^{\infty} e^{\frac{-ij\pi}{l_1}}\left\{\left(\frac{e^{\frac{A_2}{2l_2}}}{2l_2 t_2}\sum_{k=-\infty}^{\infty} e^{\frac{-ij\pi}{l_2}}\right)\eta\right\} \tag{17}$$

where

$$\eta = \tilde{f}\left[\frac{A_1}{2l_1 t_1} - \frac{ij\pi}{l_1 t_1}, \frac{A_2}{2l_2 t_2} - \frac{ij\pi}{l_2 t_2}\right]$$

From (17) we see that the two-dimensional formula is also the iterated one-dimensional formulas. In particular, if $l_1 = l_2 = 1$, then the expression within the braces in (17) is regarded as one-dimensional Euler algorithm in [1] with the one dimensional transform replaced by the two-dimensional transform $\tilde{f}$. Given the transform $C(\gamma,\lambda)$, we first compute the Fourier inverse with respect to γ numerically. Then we invert the Laplace transform with respect to λ by using the numerical univariate inversion formula.

Let L^{-1} and F^{-1} denote respectively the Laplace and Fourier inverses, then the function $c(k,h)$ gives;

$$c(k,h) = e^{a_f k}L^{-1}(F^{-1}(C(\gamma+ia_f,\lambda);\gamma\to k);\lambda\to h)$$

Also $c(k,h)$ can be defined as

$$c(k,h) := e^{a_f k}L^{-1}(F^{-1}(C(\gamma+ia_f,\lambda)))$$

Using the Fourier inversion formula, we obtain;

$$c(k,h) = e^{a_f k}L^{-1}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma k}C(\gamma+ia_f,\lambda)d\gamma\right) \tag{18}$$

Given that $|C(\gamma+ia_f,\lambda)|$ is integrable, in this case the trapezoidal rule is exact [1]. Then, if we discretize the inversion integral by a step size Δ_f, we have that:

$$c(k,h) = e^{a_f k} L^{-1}\left(\frac{\Delta_f}{2\pi}\sum_{s=-\infty}^{\infty} e^{-i\Delta_f sk} C(\Delta_f s + ia_f, \lambda)\right) \tag{19}$$

If we set $\Delta_f = \frac{\pi}{k}$ and $a_f = \frac{g_f}{2k}$, we have

$$c(k,h) = e^{0.5g_f} L^{-1}\left(\frac{1}{2k}\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k}+\frac{iA_f}{2k},\lambda\right)\right) \tag{20}$$

By substituting $\lambda = a_i + iw$ in (20) and by the means of the Bromwich contour for the inversion of the Laplace transform, where a_i is at the right of the largest singularity of the function $C(\gamma,\lambda)$. we have:

$$c(k,h) = \frac{e^{0.5g_f + a_i h}}{4\pi k i}\int_{-\infty}^{\infty} e^{\lambda h}\left(\frac{1}{2k}\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k}+\frac{ig_f}{2k}, a_i + iw\right)\right) dw \tag{21}$$

Equation (19) can be approximated again using the trapezoidal rule with step size $\Delta_f = \frac{\pi}{h}$ and by setting $a_i = \frac{g_i}{2h}$, with g_i such that a_i is greater than the right-most singularities, this yields

$$c(k,h) \approx \frac{e^{0.5(g_f+g_i)}}{4kh}\sum_{m=-\infty}^{\infty}(-1)^m\left(\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k}+\frac{ig_f}{2k}, a_i + \frac{is\pi}{h}\right)\right) \tag{22}$$

This is an inversion formula. M. Craddock [2] discussed the sources of error and how it can be controlled by g_f and g_i in (22). He also obtained through numerical experiments for these two parameters as $g_f = g_i = 18.4$. In this work Euler transformation will be used since it gives a much faster convergence for infinite sums [1, 2]. Specifically, the Euler sum provides an estimate $E(m,n)$ of the series

$$\sum_{s=1}^{\infty}(-1)^s a_s$$

with

$$E(m,n) = \sum_{j=0}^{n-1}\binom{j}{n} 2^{-n} S_{m+j} \tag{23}$$

and

$$S_i = \sum_{j=0}^{n-1}(-1)^j a_j \tag{24}$$

The use of the Euler algorithm requires $(m+n)$ evaluation of the complex function a_j. In particular, Fourier and Laplace inversions require $(m_f+n_f)(m_i+n_i)$ evaluations of the double transform. The computational cost of the inversion is directly related to this product. In order to avoid numerical difficulties in the computation of the binomial coefficient in the Euler algorithm, we set

$$n_f = m_f + 15 \tag{25}$$

$$n_i = m_i + 15 \tag{26}$$

where the choice of m_f and m_i has to be tuned according to the volatility level.

3. The Double Transform for the Lookback Options

This section presents the Laplace transform for pricing lookback options.

3.1. The Laplace Transform for the Lookback Options

In a discrete time setting the minimum(maximum) of the asset price will be determined at discrete monitoring points. We assume that the monitoring points are equally spaced in time. More precisely, consider the asset value S_t, monitored in the interval $[0,T]$ at a sequence of equally spaced monitoring points, $0 \equiv t_0 < t_1 < ... < t_m \equiv T$. Let $X_i = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$, where X_i is the return between t_{i-1} and t_i and $S_k = S_{t_k} = S_{k-1}e^{X_k} = S_0 e^{(X_1,X_2,...,X_k)}, k = 0,1,2,...,m$. We consider the maxima and minima of the asset price only at the monitoring points as,

$$M_{t_j,t_k} = \max_{j\leq l\leq k} S_l,\ 0 \leq j \leq k \leq m \tag{27}$$

$$M_{0,T} = \max_{0\leq k\leq m} S_k,\ m_{0,T} = \min_{0\leq k\leq m} S_k \tag{28}$$

Where $t_{j-1} \leq t < t_j$, standard finance theory gives the values of the floating lookback call and put options respectively at any $t \in [0,T]$ as

$$LC(t,T) = e^{-r(T-t)} E^* \left[S_T - m_{0,T} \mid F_t\right] \tag{29}$$

$$LP(t,T) = e^{-r(T-t)} E^* \left[M_{0,T} - S_T \mid F_t\right] \tag{30}$$

where r is the risk-free interest rate and E^* is the expectation under the risk-neutral measure (the measure could be specified by arbitrage arguments for the

Brownian model or by equilibrium arguments for general models). In the same way, at any time $t \in [0, T]$, for the fixed strike call and put we have respectively

$$FC(t,T) = e^{-r(T-t)} E^* \left[(M_{0,T} - K)^+ \mid F_t \right] \tag{31}$$

$$FP(t,T) = e^{-r(T-t)} E^* \left[(K - m_{0,T})^+ \mid F_t \right] \tag{32}$$

The price of a lookback put option under a given risk-neutral measure is given by

$$LP(t,T) = e^{-r(T-t)} E^* \left[(M_{0,T} - S_T) \mid F_t \right] \tag{33}$$

$$= e^{-r(T-t)} E^* \left[M_{0,T} \mid F_t \right] - S_t \tag{34}$$

To compute the value of $E^*[M_{0,T} \mid F_t]$ we consider any time $t \in [t_{j-1,t_j}]$, with $m \geq j \geq 1$. Since $\max(x_1, x_2) = \frac{(x_1+x_2)+(x_2-x_1)}{2} = x_1 + \max(x_2 - x_1, 0)$, we have that

$$E^*[M_{0,T} \mid F_t] = E^*[M_{t_j,T} \mid F_t] + E^*[(M_{0,t_{j-1}} - M_{t_j,T}) I_{\{M_{0,t_{j-1}} \geq M_{t_j,T}\}} \mid F_t] \tag{35}$$

where

$$M_{t_j,T} = \max_{j \leq l \leq m}, S_l = S_t e^{X_{t,t_j} + N_{j,m}} = S_t e^{Y^t_{j,m}}$$

and

$$A(u;t) = E^*[e^{uY^t_{j,m}}] = X_{j,m} E^*[e^{uX_{t,t_j}}], Y^t_{j,m} = X_{t,t_j} + N_{t,m} \tag{36}$$

Then from (36), (35) becomes

$$E^*[M_{t_j,T} \mid F_t] = S_t E^*[e^{Y^t_{j,m}}] = S_t A(1,t) \tag{37}$$

Since $A(1,t)$ can be computed using (36) and $X_{j,m}$, we only need to compute the second term in the right hand side of (35). If $t = t_j$ is a monitoring point and $S_{t_j} \leq M_{0,t_{j-1}}$, that is, whenever the previous maximum of the asset price is less than the value at the jth monitoring point (and can therefore be ignored), then the second term in the RHS of (35) is zero. However, in general, when either t is not a monitoring point or $t = t_j$ but $S_{t_j} < M_{0,t_{j-1}}$, it is necessary to compute the second term in the RHS of (35). For this purpose, following the Laplace transform of a call option price introduced by M. C. Fu et al [4]

$$\Phi_{0,T}(\lambda) = E\left[\int_0^\infty e^{-\lambda k} C(K,T,S,r,\sigma) dk \right]$$

The above equation can be written as

$$\Phi_{0,T}(\lambda) = e^{-rT} \left(\frac{E[R]\lambda + \psi_{0,T}(\lambda) - 1}{\lambda^2} \right)$$

where $E[R] = \frac{S(e^{rT}-1)}{r}$ and $\psi_{0,T} = E(e^{-\lambda R})$. We now derive the Laplace transform of the second term in the RHS of (35). The value of a floating strike lookback call option as

$$LC(t,T) = S_T - e^{-r(T-t)}E^*\left[m_{0,T} \mid F_t\right] \tag{38}$$

$$= S_T - e^{-r(T-t)}E^*\left[m_{0,T} \mid F_t\right] \tag{39}$$

$$= S_t - e^{-r(T-t)}\Big(E^*[m_{t_j,T} \mid F_t] + E^*[m_{t_j,T} - (m_{0,t_{j-1}})I_{\{m_{0,t_{j-1}} \leq m_{t_j,T}\}} \mid F_t]\Big) \tag{40}$$

Where $E^*[m_{t_j,T} \mid F_t] = S_tB(1,t)$.

The following results give some basic properties of the Laplace transform for pricing lookback put and call options.

Theorem 2. *Let $\omega > 1$ and assume that $A(1-\omega;t) < \infty$. At any time $t \in [t_{j-1}, t_j], m \geq j \geq 1$, using (36) the Laplace transform of*

$$f(y,S_t) = E^*\left[e^y - (M_{t_j,T})\mathbf{I}_{\{e^y \geq M_{t_j,T}\}} \mid F_t\right] \tag{41}$$

with respect to y is given by

$$h(\omega) = \int_{-\infty}^{\infty} e^{-\omega y} f(y,S_t)dy = \frac{(S_t)^{(1-\omega)}}{\omega(\omega-1)}A(1-\omega : t) \tag{42}$$

Proof. We denote the risk-neutral density of $Y_{j,m}^t$ by $\rho(Y_{j,m}^t;z)$, we can rewrite (41) as

$$f(y;S_t) = \int_{-\infty}^{y-\log S_t} (e^y - S_te^z)\rho(Y_{j,m}^t;z)dz \tag{43}$$

The Laplace transform for (43) is given by

$$h(\omega) = \int_{-\infty}^{\infty} e^{-\omega y}\left[\int_{-\infty}^{y-\log S_t} (e^y - S_te^z)\rho(Y_{j,m}^t;z)dz\right]dy \tag{44}$$

By means of Fubini's Theorem, (44) becomes

$$h(\omega) = \int_{-\infty}^{\infty}\left[\left(\int_{z+\log S_t}^{\infty} e^{(1-\omega)y}dy\right)\rho(Y_{j,m}^t;z)\right]dz$$

$$-\int_{-\infty}^{\infty}\left[\left(\int_{z+\log S_t}^{\infty} e^{-\omega y}dy\right)S_te^z\rho(Y_{j,m}^t;z)\right]dz$$

$$= \left(\frac{S_t^{(1-\omega)}}{\omega(\omega-1)} - \frac{S_t^{(1-\omega)}}{\omega} \right) \int_{-\infty}^{\infty} e^{(1-\omega)z\rho(Y_{j,m}^t;z)} dz$$

$$= \frac{S_t^{(1-\omega)}}{\omega(\omega-1)} E^*[e^{(1-\omega)Y_{j,m}^t}] \tag{45}$$

Theorem 3. *Let $\omega > 1$ and assume that $A(1-\omega; t) < \infty$. At any time $t \in [t_{j-1}, t_j], m \geq j \geq 1$, using (36) the Laplace transform with respect to y of*

$$g(y, S_t) = E^* \left[(m_{t_j,T} - e^y) \mathbf{I}_{\{e^y \leq m_{t_j,T}} \mid F_t \right] \tag{46}$$

is given by

$$d(\omega) = \frac{S_t^{(\omega+1)} E^*[e^{(\omega+1)(X_{t,t_j}+N_{j,m})}]}{\omega(\omega+1)}$$

Therefore,

$$d(\omega) = \frac{S_t^{(\omega+1)} E^*[e^{(\omega+1)H_{j,m}^t}]}{\omega(\omega+1)} \tag{47}$$

Proof. We denote the risk-neutral density of $H_{j,m}^t$ by $\rho(H_{j,m}^t; z)$, we can rewrite (46) as

$$g(y; S_t) = \int_{-\infty}^{y-\log S_t} (S_t e^z - e^y) \rho(H_{j,m}^t; z) dz \tag{48}$$

The Laplace transform for (48) is given by

$$d(\omega) = \int_{-\infty}^{\infty} e^{-\omega y} \left[\int_{-\infty}^{y-\log S_t} (S_t e^z - e^y) \rho(H_{j,m}^t; z) dz \right] dy \tag{49}$$

By means of Fubini's Theorem, (49) becomes

$$d(\omega) = \int_{-\infty}^{\infty} \left[\left(\int_{-z-\log S_t}^{\infty} e^{-\omega y} dy \right) S_t e^z \rho(Y_{j,m}^t; z) \right] dz$$

$$- \int_{-\infty}^{\infty} \left[\left(\int_{-z-\log S_t}^{\infty} e^{(1+\omega)y} dy \right) \rho(H_{j,m}^t; z) \right] dz$$

$$= \left(\frac{(S_t)^{(1+\omega)}}{\omega} - \frac{(S_t)^{(1+\omega)}}{\omega(\omega+1)} \right) \int_{-\infty}^{\infty} e^{(1+\omega)z} \rho(H_{j,m}^t; z) dz$$

$$= \frac{(S_t)^{(1+\omega)} E^*[e^{(1+\omega)H^t_{j,m}}]}{\omega(\omega+1)} \tag{50}$$

The Corollary 1 presents the three greeks (delta, gamma and vega) of the Laplace transform for pricing Lookback put option.

Corollary 1. *Laplace Transform for Pricing Lookback Put Option At any time $t_{j-1} \leq t < t_j$, with $1 \leq j \leq m$, we have*

$$LP(t,T) = e^{-r(T-t)} \left[S_t A(1,t) + L^{-1} h(\omega) \mid_{\log M_{0,t_{j-1}}} \right] - S_t \tag{51}$$

where

$$h(\omega) = \int_{-\infty}^{\infty} e^{-\omega y} f(y, S_t) dy = \frac{S_t^{(1-\omega)}}{\omega(\omega-1)} A(1-\omega : t)$$

Then from (51) we have the following greeks;

$$\Delta(LP(t,T)) = \frac{\partial LP(t,T)}{\partial S_t}$$

$$= e^{-r(T-t)} \left[A(1,t) - L^{-1} \left(\frac{S_t^{-\omega}}{\omega} A(1-\omega : t) \right) \mid_{\log M_{0,t_{j-1}}} \right] - 1 \tag{52}$$

$$\Gamma(L(P(t,T))) = \frac{\partial^2 LP(t,T)}{\partial S_t^2}$$

$$= e^{-r(T-t)} L^{-1} S_t^{-(1+\omega)} A(1-\omega; t)) \mid_{\log M_{0,t_{j-1}}} \tag{53}$$

$$VG(LP(t,T)) = \frac{\partial LP(t,T)}{\partial \sigma}$$

$$= e^{-r(T-t)} \left[S_t \frac{\partial A(1,t)}{\partial \sigma} - L^{-1} \left(\frac{\partial h(\omega)}{\partial \sigma} \right) \right] \mid_{\log M_{0,t_{j-1}}} \tag{54}$$

Similarly, the Corollary 2 presents the three greeks (delta, gamma and vega) of the Laplace transform for pricing Lookback call option.

Corollary 2. *Laplace Transform for Pricing Lookback Call Option At any time $t_{j-1} \leq t < t_j$, with $1 \geq j \geq m$, we have*

$$LC(t,T) = S_t - e^{-r(T-t)} \left[S_t B(1,t) + L^{-1} \left(\frac{S_t^{\omega+1} B(\omega+1 : t)}{\omega(\omega+1)} \right) \mid_{-\log m_{0,t_{j-1}}} \right] \tag{55}$$

Then from (55) we have the following greeks;

$$\Delta(LC(t,T)) = \frac{\partial LC(t,T)}{\partial S_t}$$

$$= 1 - e^{-r(T-t)}\left[B(1,t) - L^{-1}\left(\frac{S_t^{\omega}}{\omega}B(\omega+1:1)\right)|_{-\log m_{0,t_{j-1}}}\right] \tag{56}$$

$$\Gamma(L(C(t,T))) = \frac{\partial^2 LC(t,T)}{\partial S_t^2}$$

$$= e^{-r(T-t)}L^{-1}(S_t^{(\omega-1)}B(\omega+1;t))\,|_{-\log m_{0,t_{j-1}}} \tag{57}$$

$$VG(LC(t,T)) = \frac{\partial LC(t,T)}{\partial \sigma}$$

$$= e^{-r(T-t)}\left[S_t\frac{\partial B(\omega+1,t)}{\partial\sigma} - L^{-1}\left(\frac{S_t^{\omega+1}}{\omega(\omega+1)}\frac{\partial B(\omega+1:t)}{\partial\sigma}\right)\right]|_{-\log m_{0,t_{j-1}}} \tag{58}$$

where σ is the volatility parameter and L^{-1} is the Laplace inversion with respect to ω.

Corollaries 3 and 4 below present some special cases of the Laplace transform for the price of a fixed strike lookback put and call options.

Corollary 3. *Laplace Transform for the Price of a Fixed Strike Lookback Put Option*
At any time $t_{j-1} \le t < t_j, m \le j \le 1$, the price of a fixed strike lookback put option is given by

$$FP(t,T) = \left.\begin{array}{r} Ke^{-r(T-t)} + LC(t,T) - S_t, \text{ if } m_{0,t_{j-1}} \le K \\ e^{-r(T-t)}[K - S_tB(1;t) + g(-\log K, S_t)], \text{ if } m_{0,t_{j-1}} > K \end{array}\right\} \tag{59}$$

Corollary 4. *Laplace Transform for the Price of a Fixed Strike Lookback Call Option*
At any time $t_{j-1} \le t < t_j, j \le 1$, the price of a fixed strike lookback call option is given by

$$FC(t,T) = \left.\begin{array}{r} -Ke^{-r(T-t)} + LP(t,T) + S_t, \text{ if } M_{0,t_{j-1}} \le K \\ e^{-r(T-t)}[-K + S_tA(1;t) + f(-\log K, S_t)], \text{ if } M_{0,t_{j-1}} > K \end{array}\right\} \tag{60}$$

3.2. Numerical Inversion for Lookback Option

The Euler algorithm has gained a lot of popularity in queueing and network analysis due to its simplicity of implementation, speed and high accuracy. In finance, M. C Madan et al [4] used it to price continuous Asian options by inverting the Laplace transform of the form

$$C_{t,T}(K) = \frac{4S_t e^{-r(T-t)} C^{(\nu)}(h,q)}{\sigma^2 T} \tag{61}$$

given by [6], where

$$\nu = \frac{2r}{\sigma^2 - 1}, h = \frac{\sigma^2(T-t)}{4}, q = \frac{\sigma^2(KT - ta_t)}{4S_t} \text{ and } a_t = \frac{\int_0^t S_u du}{t}$$

The average to time t is denoted by a_t, and the Laplace transform of $C^{(\nu)}(.,q)$ in the first parameter is given by

$$\hat{C}(\lambda, q) = \int_0^t e^{-\lambda h} C^{(\nu)}(h,q) dh \tag{62}$$

We present below the Laplace inversion for floating lookback call and put options.

3.2.1. Laplace Inversion for Floating Lookback Put Option

Consider the Laplace transform of the form

$$f(M_{0,t_{j-1}} : S_t) = E^*[(M_{0,t_{j-1}} - M_{t_j,T})\mathbf{I}_{\{M_{0,t_{j-1}} \geq M_{t_j,T}\}} \mid F_t] \tag{63}$$

We rescale the constant $C \leq \min(S_t, M_{0,t_{j-1}})$ to get

$$f(M_{0,t_{j-1}} : S_t) = CE^*\left[\left(\frac{M_{0,t_{j-1}} - S_t e^{Y^t_{j,m}}}{C}\right)\mathbf{I}_{\{Y^t_{j,m} \leq \log M_{0,t_{j-1}} - \log S_t\}} \mid F_t\right] \tag{64}$$

The purpose of introducing the arbitrary constant, C is to make sure that the Laplace inversion will not be evaluated at extreme points for a wide range of model parameters. When t is a monitoring point, we set $C = \min(S_t, 0.99M_{0,t_{j-1}})$. Thus the function $f(N, S_t)$ is confined to the positive real line, because $N = \log\left(\frac{M_{0,t_{j-1}}}{C}\right) \geq 0$. We refer to the one-sided version (one dimension) of the Euler algorithm. Using $N_1 = N_2 = 40$ iterations to compute the partial sums ensures extremely accurate results. If t is not a monitoring point with $t_{j-1} \leq t < t_j$, we have that $f(M_{0,t_{j-1}}; S_t \leq Ce^N P(X_{t,t_j} \leq N))$, since $Y^t_{j,m} \leq X_{t,t_j}$ and $\log\left(\frac{S_t}{C}\right) \geq 0$, where $N = \log \frac{M_{0,t_{j-1}}}{C}$. We can then apply the results in [12] for a plain vanilla put option. We choose $N = \beta A$ with

$$\beta = \min\left(\sigma\sqrt{t_j - t}, \frac{1}{4}\right), \quad A = 18.4 \tag{65}$$

if the resulting C satisfies $C \leq S_t$, which holds in all numerical cases. In the double exponential jump diffusion model we need to be sure the characteristic function of X_{t,t_j} has no point of catastrophe, to avoid discontinuity we set

$$\beta = \max\left(\frac{1}{\alpha_2}, \min\left(\sigma\sqrt{t_j - t}, \frac{1}{4}\right)\right) \tag{66}$$

We set the number of iterations to $N_1 = N_2 = 150$ in all cases to ensure high accuracy in the inversion.

3.2.2. Laplace Inversion for Floating Lookback Call Option

Similarly, we are going to invert the Laplace transform of the form

$$g(m_{0,t_{j-1}} : S_t) = E^*[(m_{0,t_{j-1}} - m_{t_j,T})\mathbf{I}_{\{M_{0,t_{j-1}} \leq M_{t_j,T}\}} \mid F_t] \tag{67}$$

We rescale the constant C to get

$$g(m_{0,t_{j-1}} : S_t) = \frac{1}{C}E^*\left[\left(CS_t e^{H^t_{j,m}} - Cm_{0,t_{j-1}}\right)\mathbf{I}_{\{H^t_{j,m} \geq \log m_{0,t_{j-1}} - \log S_t\}} \mid F_t\right] \tag{68}$$

When t is a monitoring point, we set $C = \min(\frac{1}{S_t}, \frac{0.99}{m_{0,t_{j-1}}})$. Thus the function $g(N, S_t)$ is confined to the positive real line, because $n = -\log\left(Cm_{0,t_{j-1}}\right) \geq 0$. We refer to the one-sided version of the Euler algorithm. Using $N_1 = N_2 = 40$ iterations to compute the partial sums ensures extremely accurate results. If t is not monitoring point with $H^t_{j,m} < X_{t,t_j}$, we have $g(n; S_t \leq S_t E^*\left[(e^{X_{t,t_j}} - e^{-n})\mathbf{I}_{\{X_{t,t_j} - n\}}\right]$. We can then apply the results in [12] for a plain vanilla put option. We choose $n = \beta B$ with

$$\beta = \min\left(\sigma\sqrt{t_j - t}, \frac{1}{4}\right), \quad B = 18.4 \tag{69}$$

In the double exponential jump diffusion model we need to be sure that the characteristic function of X_{t,t_j} has no singular point, to avoid this we set

$$\beta = \max\left(\frac{1}{\alpha_1 - 2}, \min\left(\sigma\sqrt{t_j - t}, \frac{1}{4}\right)\right) \tag{70}$$

We set the number of iterations to $N_1 = N_2 = 150$ in all cases to ensure high accuracy in the inversion.

4. Numerical Experiments

This section presents some numerical experiments of Laplace transform for pricing Asian and lookback options

Example 1: We consider the pricing of Asian option using the following parameters:

$$S_0 = 100, \sigma = \frac{10}{100}, \frac{20}{100}, \frac{30}{100}, \frac{40}{100}, K = 90, 95, 100, r = \frac{9}{100}, T = 1$$

and

$$m_i = 135, n_i = m_i + 15, m_f = 55, n_f = m_f + 15, g_i = g_f = 22.4$$

The parameter settings and the results generated are shown in Tables 1 and 2 below.

Example 2: We compare the performance of Laplace transform for pricing Asian call option with other methods such as [1], [10], [15] and [18] with the parameters: $K = 2, t = 0$, where the values of r, σ and T vary. The comparative result analysis is shown in Table 4 below.

Example 3: We consider the pricing of floating lookback put option under double exponential jump diffusion model with the parameters

$$S_0 = 100, r = 0.1, \sigma = 0.212, \lambda = 2.29, \alpha_1 = 10, \alpha_2 = 5.71, T = 0.5, p = 0.6$$

The comparative results analysis of Laplace transform and Monte Carlo methods for $M = 110$ and $M = 120$ are shown in Tables 5 and 6 below respectively.

4.1. Discussion of the Results

In Tables 1, 2 and 3, we considered how to select the values of m_f and m_i in the Euler algorithm in order to achieve a given accuracy and how they affect the estimate in the pricing of Asian option. We can also see from the Tables 1 and 2 below that as the volatility increases, the optimal values of m_f and m_i decrease quickly and consequently the computational time required for estimating the option price decreases. In Table 3, we compared the Laplace transform method with other approximations namely lognormal density derived by E. Levy [11], Crank Nicolson finite difference method with 3000 spatial and time grids. Also Crank Nicolson Finite difference method with upper bound by L. C. G. Rogers and et al [13], we can also see that the Laplace transform performs better in pricing Asian Options. From Table 4, the range of prices obtained with the Laplace transform inversion method for pricing Asian call option are very close to outputs given by the Euler numerical approach. From Tables 5 and 6, we can see that Laplace transform performs better than its counterpart Monte Carlo method when pricing the floating lookback put option. Our numerical analysis demonstrates that the Laplace Transform is accurate and perform very well for pricing path dependent options.

5. Conclusion

In this paper, we have considered a performance measure of Laplace transform for pricing path dependent options namely "Asian and Lookback options" against some existing approximation techniques. We also discussed the numerical inversion and obtained very accurate results, in particular for the difficult volatility levels. The outputs from our algorithm present accurate results in comparison to different numerical approximation methods available in the literatures and the numerical inversion in [6], Laplace transform seems likely to be quick if the implemented initial parameters do not take extreme values. One limitation of the Laplace transform for pricing lookback option is that the European call and put prices have to be computed accurately and fast, preferably by using analytical formula to reduce errors and to increase the speed in computing the recursions. The method could also be extended for pricing other derivatives, whose values are function of the joint distribution of the terminal asset value and its discretely monitored maximum (or minimum) throughout the lifetime of the option, such as partial lookback options. The method of Laplace transform leads to good results and is moreover less time consuming than the Euler method for example.

Appendices

No. of Decimal Digits	**2**	**3**	**4**	**5**
Volatility, σ	$m_f; m_i$	$m_f; m_i$	$m_f; m_i$	$m_f; m_i$
0.1	15;115	15;115	35;115	35;135
0.2	15;15	15;35	15;55	15;55
0.3	15;15	15;35	15;15	15;15
0.4	15;15	15;15	15;15	15;15

Table 1: Accuracy Desired and Parameters of the Euler Algorithm

$m_f; m_i$	σ	15	35	55	75	95	115	135
15	0.10	5.293	4.913	4.904	4.913	4.915	4.915	4.915
35	0.10	5.293	4.913	4.904	4.913	4.915	4.915	4.915
55	0.10	5.293	4.913	4.904	4.913	4.915	4.915	4.915
75	0.10	5.293	4.913	4.904	4.913	4.915	4.915	4.915
95	0.10	5.293	4.913	4.904	4.913	4.915	4.915	4.915
15	0.20	6.776	6.777	6.777	6.777	6.777	6.777	6.777
35	0.20	6.776	6.777	6.777	6.777	6.777	6.777	6.777
55	0.20	6.776	6.777	6.777	6.777	6.777	6.777	6.777
75	0.20	6.776	6.777	6.777	6.777	6.777	6.777	6.777
95	0.20	6.776	6.777	6.777	6.777	6.777	6.777	6.777
15	0.30	8.828	8.829	8.829	8.829	8.829	8.829	8.829
35	0.30	8.828	8.829	8.829	8.829	8.829	8.829	8.829
55	0.30	8.828	8.829	8.829	8.829	8.829	8.829	8.829
75	0.30	8.828	8.829	8.829	8.829	8.829	8.829	8.829
95	0.30	8.828	8.829	8.829	8.829	8.829	8.829	8.829
15	0.40	10.924	10.924	10.924	10.924	10.924	10.924	10.924
35	0.40	10.924	10.924	10.924	10.924	10.924	10.924	10.924
55	0.40	10.924	10.924	10.924	10.924	10.924	10.924	10.924
75	0.40	10.924	10.924	10.924	10.924	10.924	10.924	10.924
95	0.40	10.924	10.924	10.924	10.924	10.924	10.924	10.924

Table 2: The Parameters of the Euler Algorithm and Asian Option Prices

σ	K	**Lognormal**	**Crank Nicolson** with 3000 spatial and time grids	**Double N. Inversion** $n_f; m_f = 55; 70$**,** $n_i; m_i = 175; 190$
0.1	90	13.386	13.385	13.385
0.1	95	8.917	8.910	8.912
0.1	100	4.909	4.913	4.915
0.2	90	13.862	13.831	13.832
0.2	95	10.030	9.996	9.996
0.2	100	6.804	6.777	6.777
0.3	90	15.067	14.984	14.984
0.3	95	11.733	11.656	11.656
0.3	100	8.886	8.829	8.829
0.4	90	16.654	16.500	16.500
0.4	95	13.648	13.511	13.524
0.4	100	11.031	10.923	10.924

Table 3: The Comparative Result Analysis of Asian Option Pricing Models

r	σ	T	S	**LT**	**Euler**	**Shaw**	**TW**	**MCM**	**Std err**
0.0125	0.25	2	2.0	0.171	0.172	0.172	0.173	0.172	(0.003)
0.02	0.1	1	2.0	0.058	0.056	0.520	0.057	0.057	(0.001)
0.05	0.5	1	1.9	0.194	0.194	0.193	0.195	0.196	(0.004)
0.05	0.5	1	2.0	0.247	0.247	0.246	0.250	0.249	(0.004)
0.05	0.5	1	2.1	0.303	0.307	0.306	0.311	0.309	(0.005)
0.05	0.5	2	2.0	0.346	0.352	0.350	0.359	0.348	(0.007)
0.18	0.3	1	2.0	0.218	0.219	0.217	0.220	0.220	(0.003)

Table 4: The Comparative Result Analysis of Laplace Transform Inversion (LT) for Pricing Asian Options with other Methods: Euler [1], Shaw [15], Turnbull-Wakeman, (TW) [18] and Monte Carlo Method, (MCM) [10].

Points (m)	**Price LT**	**MCM (Std err)**	**Δ: LT**	**Δ: MCM (Std err)**	**Γ: LT**	**Γ: MCM (Std err)**
5	13.634	13.626	-0.372	-0.372	0.031	0.031
		(0.0051)		(0.0002)		(0.0002)
10	14.285	14.279	-0.329	-0.330	0.033	0.033
		(0.0052)		(0.0002)		(0.0002)
20	14.802	14.791	-0.297	-0.298	0.034	0.033
		(0.0053)		(0.0002)		(0.0002)
40	15.194	18.188	-0.274	-0.274	0.034	0.034
		(0.0054)		(0.0002)		(0.0002)
80	15.482	15.476	-0.257	-0.257	0.035	0.035
		(0.0054)		(0.0002)		(0.0002)
160	15.693	15.690	-0.244	-0.245	0.035	0.036
		(0.0055)		(0.0002)		(0.0002)

Table 5: The Comparison of the Laplace Transform, (LT) and Monte Carlo Method, (MCM) for pricing Floating Lookback Put Option with $M = 110$

Points (m)	**Price LT**	**MCM (Std err)**	**Δ: LT**	**Δ: MCM (Std err)**	**Γ: LT**	**Γ: MCM (Std err)**
5	19.370	19.364	-0.609	-0.372	0.023	0.023
		(0.0042)		(0.0002)		(0.0002)
10	19.751	19.751	-0.580	-0.581	0.025	0.025
		(0.0044)		(0.0002)		(0.0002)
20	20.067	20.060	-0.558	-0.588	0.026	0.026
		(0.0045)		(0.0002)		(0.0002)
40	20.309	20.305	-0.541	-0.541	0.027	0.027
		(0.0046)		(0.0002)		(0.0002)
80	20.488	20.484	-0.528	-0.529	0.027	0.027
		(0.0046)		(0.0002)		(0.0002)
160	20.621	20.620	-0.519	-0.519	0.028	0.028
		(0.0047)		(0.0002)		(0.0002)

Table 6: The Comparison of the Laplace Transform (LT) and Monte Carlo Method, (MCM) for Pricing Floating Lookback Put Option with $M = 120$

The above results can be obtained using Mathematica.

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