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POSITIVE SOLUTION FOR A NONLINEAR FOURTH ORDER PERIODIC BOUNDARY VALUE PROBLEM

Huijun Zheng
School of Mathematica Science and Technology
Northeast Petroleum University
Daqing, 163318, P.R. CHINA

Abstract: The nonlinear fourth order periodic boundary value problem is studied.The existence of positive solution solution is proved under appropriate conditions by using the fixed point index and the property of Green's function.

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1. Introduction

In this paper,we are concerned with the following nonlinear fourth order periodic boundary value problem (BVP)

$$u^{(4)}(t) + \rho^4 u(t) = f(t, u(t)),\ 0 < t < 2\pi \tag{1}$$

$$u^{(i)}(0) = u^{(i)}(2\pi),\ i = 0, 1, 2, 3 \tag{2}$$

where $\rho \in (0, \frac{1}{2\sqrt{2}})$, $f : [0, 2\pi] \times [0, +\infty) \to [0, +\infty)$ is a nonnegative continuous function and $0 < \int_0^{2\pi} f(t, u(t))\mathrm{d}t < +\infty$. By a positive solution of the periodic BVP (1), (2), we mean a function $u(t) > 0$ for $t \in (0, 2\pi)$ and $u(t) \in C^2[0, 2\pi] \cap C^3(0, 2\pi)$ such that $u(t)$ satisfied differential equation (1) and the boundary conditions (2). It is assumed throughout that:

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(H_1) Function $f(t,u)$ is continuous and nonnegative in the interval

$$[0,2\pi]\times[0,+\infty),\ \text{and } 0<\int_0^{2\pi} f(t,u)dt<+\infty;$$

(H_2) $\lim\limits_{u\to 0}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}<\rho^4,\ \lim\limits_{u\to+\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}>\rho^4;$

(H_3) $\lim\limits_{u\to 0}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}>\rho^4,\ \lim\limits_{u\to+\infty}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}<\rho^4.$

The nonlinear fourth order periodic boundary problem arises in many branches of applied mathematics and physics, for detail, see[1-6] and the references therein. In[1], Cabada and Lois obtained maximum principles for the fourth-order operator $T_4[M]u(t)=u^{(4)}(t)+Mu(t)$ under the periodic boundary condition and proved the operator $T_4[M]$ is inverse positive on

$F_4=\{u\in W^{4,1}(I): u^{(i)}(0)=u^{(i)}(2\pi), i=1,2; u^{(3)}(0)\geq u^{(1)}(1)\}$ with $I=[0,2\pi]$ if $M\in(0,M_4^4]$, where $M_4=1.0646$ is the unique solution of the equation $\tan\pi\frac{\sqrt{2}}{2}m=-\tanh\pi\frac{\sqrt{2}}{2}m$, in the interval $(\frac{\sqrt{2}}{2},\sqrt{2})$.Further they proved the existence of solution and validity of the monotone method in the presence of lower and upper solutions for the periodic boundary value problem

$$u^4(t)=f(t,u(t)),\ 0\leq t\leq 1, \tag{3}$$

$$u^{(i)}(0)=u^{(i)}(2\pi),\ i=0,1,2,3. \tag{4}$$

Recently, Li[2] has dealt with a maximum principle for $L_4u=u^{(4)}-\beta u''+\alpha u$ in periodic boundary condition, and has showed if α,β satisfy the assumption $\alpha>0, \sqrt{\alpha}<\frac{\beta}{2}+2\pi^2,\ \frac{\alpha}{\pi^4}+\frac{\beta}{\pi^2}+1>0$,then L_4 is strongly inverse positive in space

$E_4=\{u\in C^4[0,1]|u^{(i)}(0)=u^{(i)}(1), i=1,2; u^{(3)}(0)\geq u^{(1)}(1)\}$.

The main purpose of this paper is to discuss the existence of positive solutions to the periodic BVP (1), (2), the method applied is different from that of [2], in fact, by employing the positivity of Green's function and the fixed point index in cone, we establish the existence of positive solution to the periodic BVP (1), (2) if the nonlinearity f satisfy (H_1),(H_2).

2. Preliminaries and Lemmas

Consider the Banach space $C[0,2\pi]$ with norm $||u||=\sup\limits_{t\in[0,2\pi]}|u(t)|$ and let

$$C^+[0,2\pi]=\{u\in C[0,2\pi]; u(t)\geq 0, 0\leq t\leq 2\pi\},$$

$$K = \{u \in C^+[0, 2\pi]; \min_{t\in[0,2\pi]} u(t) \geq \sigma||u||\}$$

where $\sigma = \frac{m^2}{M^2}$ and

$$m = \beta e^{-\sqrt{2}\rho\pi} \sin\sqrt{2}\rho\pi, \quad M = \beta(e^{\sqrt{2}\rho\pi} + 1),$$

$$\beta = \frac{\sqrt{2}}{\rho(e^{\sqrt{2}\rho\pi} - 2\cos\sqrt{2}\rho\pi + e^{-\sqrt{2}\rho\pi})}.$$

It is easy to know that K is a cone of nonnegative function in $C[0, 2\pi]$. A direct check implies that the BVP (1),(2) is equivalent to the following integral equation

$$u(t) = \int_0^{2\pi}\int_0^{2\pi} G_1(t, s)G_2(s, \tau)f(\tau, u(\tau))d\tau ds \quad (5)$$

where

$$G_1(t, s) = \begin{cases} \beta e^{-\frac{\sqrt{2}}{2}\rho(2\pi-t+s)}[\sin\frac{\sqrt{2}}{2}\rho(t-s) + e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(2\pi - t + s)], 0 \leq s \leq t \leq 2\pi \\ \beta e^{-\frac{\sqrt{2}}{2}\rho(s-t)}[\sin\frac{\sqrt{2}}{2}\rho(2\pi + t - s) + e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(s - t)], 0 \leq t \leq s \leq 2\pi \end{cases}$$

$$G_2(t, s) = \begin{cases} \beta e^{-\frac{\sqrt{2}}{2}\rho(t-s)}[e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(t-s) + \sin\frac{\sqrt{2}}{2}\rho(2\pi - t + s)], 0 \leq s \leq t \leq 2\pi \\ \beta e^{-\frac{\sqrt{2}}{2}\rho(2\pi+t-s)}[e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(2\pi + t - s) + \sin\frac{\sqrt{2}}{2}\rho(s - t)], 0 \leq t \leq s \leq 2\pi \end{cases}$$

Lemma 1. *For all $(s, t) \in [0, 2\pi] \times [0, 2\pi]$, we have $m \leq G_i(t, s) \leq M$, $i = 1, 2$.*

Proof. Let $h(t) = \sin\frac{\sqrt{2}}{2}\rho t + e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(2\pi - t)$, then $h(0) = e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi$ and $h(2\pi) = \sin\sqrt{2}\rho\pi$. It is easy to see

$$h'(t) = \frac{\sqrt{2}}{2}\rho[\cos\frac{\sqrt{2}}{2}\rho t - e^{\sqrt{2}\rho\pi}\cos\frac{\sqrt{2}}{2}\rho(2\pi - t)],$$

$$h''(t) = -\frac{\rho^2}{2}[\sin\frac{\sqrt{2}}{2}\rho t + e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(2\pi - t)] < 0$$

this shows that $h'(t)$ is decreasing on $[0, 2\pi]$, which implies $h'(t) \leq h'(0) = \frac{\sqrt{2}}{2}\rho(1 - e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi)$.

If $e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi \geq 1$, then $h'(t) \leq 0$ and hence $h(2\pi) \leq h(t) \leq h(0)$, thus, we have $\sin\sqrt{2}\rho\pi \leq h(t) \leq e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi$.

If $e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi < 1$, then $h''(t) < 0$ and hence we know that $h(t)$ is convex function on $[0, 2\pi]$. Let $h'(t) = 0$, we obtain $\tan\frac{\sqrt{2}}{2}\rho t = \frac{1-e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi}{e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi}$, and hence

$t_0 = \frac{\sqrt{2}}{\rho}[\arctan(\frac{1-e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi}{e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi}) + k\pi],\ k = 0, \pm1, \pm2, \ldots\ldots$

It follows from $\frac{\sqrt{2}}{2}\rho t_0 \in [0, \frac{\pi}{2})$ that we know that $k = 0$, and hence the unique critical point $t_0 = \frac{\sqrt{2}}{\rho}\arctan(\frac{1-e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi}{e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi})$ of $h(t)$ on $[0, 2\pi]$ and the critical value

$h(t_0) = \sin\frac{\sqrt{2}}{2}\rho t_0 + e^{\sqrt{2}\rho\pi}\sin\frac{\sqrt{2}}{2}\rho(2\pi - t_0)$

$$= e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi\cos\frac{\sqrt{2}}{2}\rho t_0[1 + \frac{1-e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi}{e^{\sqrt{2}\rho\pi}\sin\sqrt{2}\rho\pi}\tan\frac{\sqrt{2}}{2}\rho t_0]$$

$$= \sqrt{e^{2\sqrt{2}\rho\pi} - 2e^{\sqrt{2}\rho\pi}\cos\sqrt{2}\rho\pi + 1}$$

thus, we obtain $h(t) \leq \max\{h(0), h(2\pi), h(t_0)\} \leq e^{\sqrt{2}\rho\pi} + 1$. In addition,we also have $h(t) \geq \min\{h(0), h(2\pi), h(t_0)\} \geq \sin\sqrt{2}\rho\pi$.

Consequently, we have $\sin\sqrt{2}\rho\pi \leq h(t) \leq e^{\sqrt{2}\rho\pi} + 1$.Further, we obtain $m \leq G_1(t,s) \leq M$.In very much the same way, we can prove $m \leq G_2(t,s) \leq M$.

Define the mapping $\Phi : C^+[0, 2\pi] \to C^+[0, 2\pi]$ by

$$(\Phi u)(t) = \int_0^{2\pi}\int_0^{2\pi} G_1(t,s)G_2(s,\tau)f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s \tag{6}$$

then, only and if only the nonzero fixed point $u(t)$ of the mapping Φ defined by (6) is a positive solution of the BVP (1),(2).

Lemma 2. *$\Phi : K \to K$ is a completely continuous mapping.*

Proof For any $u \in K$, from lemma1 we have following inequality

$$\min_{t\in[0,2\pi]}(\Phi u)(t) \geq m^2\int_0^{2\pi}\int_0^{2\pi} f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s \geq \sigma||\Phi u||,$$

this shows that $\Phi(K) \subset K$.In addition,it is proved easily that $\Phi : K \to K$ is a completely continuous mapping.

The following the lemma is needed in our argument, see[6].

Lemma 3. *Let E be a Banach space and $K \subset E$ a cone in E, $K_r = \{u \in K; ||u|| < r\}$, and let $\Phi : K \to K$ is a completely continuous operator.*

(i) If $\lambda\Phi u \neq u$ for any $u \in \partial K_r$ and $0 < \lambda \leq 1$, then $i(\Phi, K_r, K) = 1$;

(ii) If there exist a $e \in K\backslash\{\theta\}$ such that $u - \Phi u \neq \tau e$ for any $u \in \partial K_r$ and $\tau \geq 0$, then $i(\Phi, K_r, K) = 0$.

3. Main Results

Theorem 1. *Assume that (H_1),(H_2) hold, then the problem(1) and (2) has at least one positive solution.*

Proof. It follows from (H_2) that we may choose $\varepsilon \in (0, \rho^4)$ and $r > 0$ such that $f(t, u) \leq (\rho^4 - \varepsilon)u,\ \forall t \in [0, 2\pi],\ 0 < u \leq r$.

Let $K_r = \{u \in K; ||u|| < r\}$, now we show that $\lambda\Phi u \neq u$ for any $u \in \partial K_r$ and $0 < \lambda \leq 1$.In fact, if there exist a $u_0 \in \partial K_r$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0\Phi u_0 = u_0$, then by definition of Φ, we know that $u_0(t)$ satisfies the BVP

$$u_0^{(4)}(t) + \rho^4 u_0(t) = \lambda_0 f(t, u_0(t)),\ 0 \leq t \leq 2\pi \tag{7}$$

$$u^{(i)}(0) = u^{(i)}(2\pi),\ i = 0, 1, 2, 3. \tag{8}$$

Integrating (7) on $[0, 2\pi]$, and employing (8) we obtain

$$\rho^4 \int_0^{2\pi} u_0(t)\mathrm{d}t = \lambda_0 \int_0^{2\pi} f(t, u_0(t))\mathrm{d}t \leq (\rho^4 - \varepsilon) \int_0^{2\pi} u_0(t)\mathrm{d}t,$$

Since $u \in K$ we know that $u_0(t) \geq \sigma||u_0|| = \sigma r > 0$ and hence $\int_0^{2\pi} u_0(t)\mathrm{d}t > 0$, further, we obtain $\rho^4 \leq \rho^4 - \varepsilon$, which is a contradiction. By Lemma3 we have

$$i(\Phi, K_r, K) = 1. \tag{9}$$

It follows again from (H_2) that there exist $\varepsilon > 0$ and $H > 0$ such that

$$f(t, u) \geq (\rho^4 + \varepsilon)u, \forall t \in [0, 2\pi], u \geq H,$$

let $C = \max\limits_{t\in[0,2\pi],u\in[0,H]} \mid f(t, u) - (\rho^4 + \varepsilon)u \mid$, then it is obvious to see

$$f(t, u) \geq (\rho^4 + \varepsilon)u - C,\ \forall t \in [0, 2\pi], u \geq 0.$$

Let $R > \max\{r, \frac{c}{\sigma\varepsilon}, H\}$ and choose $e \equiv 1$, next we show that $u - \Phi u \neq \tau e$ for any $u \in \partial K_R$ and $\tau \geq 0$. In fact, if there exist $u_0 \in \partial K_R$ and $\tau_0 \geq 0$ such that $u_0 - \Phi u_0 = \tau_0$, then by definition of Φ, we know that $u_0(t)$ satisfies the BVP

$$u_0^{(4)}(t) + \rho^4 u_0(t) = f(t, u_0(t)) + (1 + \rho^4)\tau_0,\ 0 \leq t \leq 2\pi \tag{10}$$

$$u^{(i)}(0) = u^{(i)}(2\pi),\ i = 0,1,2,3 \tag{11}$$

Integrating (10) on $[0, 2\pi]$, and employing (11) we get

$$\rho^4 \int_0^{2\pi} u_0(t)\mathrm{d}t = \int_0^{2\pi} f(t, u_0(t))\mathrm{d}t + \tau_0(1+\rho^4)\int_0^{2\pi} \mathrm{d}t$$

$$\geq (\rho^4 + \varepsilon)\int_0^{2\pi} u_0(t)\mathrm{d}t - 2\pi c,$$

and hence $\int_0^{2\pi} u_0(t)\mathrm{d}t \leq \frac{2\pi c}{\varepsilon}$. Since $\int_0^{2\pi} u_0(t)\mathrm{d}t \geq 2\pi\sigma||u_0||$,we have $||u_0|| \leq \frac{c}{\sigma\varepsilon} < R$, this contradicts with $||u_0|| = R$.By Lemma3 we know

$$i(\Phi, K_R, K) = 0. \tag{12}$$

Now by the additivity of fixed point index,(9) and (12), we have

$$i(\Phi, K_R \backslash \bar{\bar{K}}_r, K) = i(\Phi, K_R, K) - i(\Phi, K_r, K) = -1,$$

this shows that Φ has a fixed point $u(t)$ in $K_R \backslash \bar{\bar{K}}_r$, which is the positive solution of the integral equation(5), and hence $u(t)$ is the positive solution of the BVP (1),(2).

Theorem 2. *Assume that* (H_1),(H_3) *hold, then the problem*(1) *and* (2) *has at least one positive solution.*

Proof. It follows from (H_3) that there exist $\varepsilon > 0$ and $r > 0$ such that $f(t, u) \geq (\rho^4 + \varepsilon)u,\ \forall t \in [0, 2\pi],\ 0 < u \leq r$. choose $e \equiv 1$, now we prove that $u - \Phi u \neq \tau$ for any $u \in \partial K_r$ and $\tau \geq 0$. In fact, if there exist a $u_0 \in \partial K_r$ and $\tau_0 \geq 0$ such that $u_0 - \Phi u_0 = \tau_0$, then by definition of Φ, we know that $u_0(t)$ satisfies the BVP (10),(11), Integrating (10) on $[0, 2\pi]$ and employing (11), we can obtain

$$\rho^4 \int_0^{2\pi} u_0(t)\mathrm{d}t = \int_0^{2\pi} f(t, u_0(t))\mathrm{d}t + \tau_0(1+\rho^4)\int_0^{2\pi} \mathrm{d}t \geq (\rho^4 + \varepsilon)\int_0^{2\pi} u_0(t)\mathrm{d}t,$$

this shows that $\rho^4 \geq \rho^4 + \varepsilon$, which is a contradiction. By Lemma3 we have

$$i(\Phi, K_r, K) = 0. \tag{13}$$

Again by (H_3), there exist $\varepsilon \in (0, \rho^4)$ and $H > 0$ such that

$$f(t, u) \leq (\rho^4 - \varepsilon)u,\ \forall t \in [0, 2\pi],\ u \geq H.$$

let $C = \max_{t\in[0,2\pi],u\in[0,H]} \mid f(t,u) - (\rho^4 - \varepsilon)u \mid$, then it is easy to know

$$f(t,u) \leq (\rho^4 - \varepsilon)u + c, \forall t \in [0,2\pi], u \geq 0.$$

choose $R > \max\{r, \frac{c}{\sigma\varepsilon}, H\}$, now we prove that $\lambda\Phi u \neq u$ for any $u \in \partial K_R$ and $0 < \lambda \leq 1$. In fact, if there exist a $u_0 \in \partial K_R$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0 \Phi u_0 = u_0$, then by definition of Φ, we know that $u_0(t)$ satisfies the BVP (7), (8), Integrating (7) on $[0,2\pi]$ and employing (8) we obtain

$$\rho^4 \int_0^{2\pi} u_0(t)\mathrm{d}t = \lambda_0 \int_0^{2\pi} f(t,u_0(t))\mathrm{d}t \leq (\rho^4 - \varepsilon) \int_0^{2\pi} u_0(t)\mathrm{d}t + 2\pi c,$$

and hence $\int_0^{2\pi} u_0(t)\mathrm{d}t \leq \frac{2\pi c}{\varepsilon}$. In addition, we have $\int_0^{2\pi} u_0(t)\mathrm{d}t \geq 2\pi\sigma||u_0||$, moreover we obtain $||u_0|| \leq \frac{c}{\sigma\varepsilon} < R$, this contradicts with $||u_0|| = R$. By Lemma3 we know

$$i(\Phi, K_R, K) = 1. \tag{14}$$

Now by the additivity of fixed point index, (13) and (14), we have

$$i(\Phi, K_R \backslash \bar{K_r}, K) = i(\Phi, K_R, K) - i(\Phi, K_r, K) = 1,$$

this shows that Φ has a fixed point $u(t)$ in $K_R \backslash \bar{K_r}$, which is the positive solution of the integral equation(5), and hence $u(t)$ is the positive solution of the BVP (1),(2).

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