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SOME NEW SEPARATION AXIOMS VIA b-$\mathcal{I}$-OPEN SETS

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Abstract: In this paper, b-$\mathcal{I}$-open sets are used to define and study some weak separation axioms in ideal topological spaces. The implications of these axioms among themselves and with the known axioms are investigated.

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1. Introduction

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [8] and Vaidyanathasamy [10]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and

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$B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [10] of A with respect to τ and $\mathcal{I}$, is defined as follows: For $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every open neighbourhood U of $x\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ where there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In this paper, b-$\mathcal{I}$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

2. Preliminaries

Let A be a subset of a topological space (X, τ). We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is called b-open [1] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. This notion has been studied extensively in recent years by many topologists [see [4, 5, 9]] because b-open sets are only natural generalization of open sets. More importantly, they also suggest several new properties of topological spaces. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is called b-$\mathcal{I}$-open [6] if $S \subset \mathrm{Int}(\mathrm{Cl}^\star(S)) \cup \mathrm{Cl}^\star(\mathrm{Int}(S))$. The complement of a b-$\mathcal{I}$-open set is called a b-$\mathcal{I}$-closed set [6]. The intersection of all b-$\mathcal{I}$-closed sets containing S is called the b-$\mathcal{I}$-closure of S and is denoted by $b\mathcal{I}\,\mathrm{Cl}(S)$. The b-$\mathcal{I}$-Interior of S is defined by the union of all b-$\mathcal{I}$-open sets contained in S and is denoted by $b\mathcal{I}\,\mathrm{Int}(S)$. The set of all b-$\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$ is denoted by $B\mathcal{I}O(X)$. The set of all b-$\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $B\mathcal{I}O(X, x)$.

Definition 1. A function f: $(X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be b-$\mathcal{I}$-continuous [6] (resp. b-$\mathcal{I}$-irresolute) if the inverse image of every open (resp. b-$\mathcal{J}$-open) set in Y is b-$\mathcal{I}$-open in X.

Definition 2. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-regular if for each closed set F of X and each point $x \in X \backslash F$, there exist disjoint b-$\mathcal{I}$-open sets U and V such that $F \subset U$ and $x \in V$.

Definition 3. A topological space (X, τ) is said to be:

1. b-T_0 [4] if to each pair of distinct points x, y of X there exists a b-open set A containing x but not y or a b-open set B containing y but not x.

2. b-T_1 [4] if to each pair of distinct points x, y of X, there exists a pair of b-open sets, one containing x but not y and the other containing y but not x.

3. b-T_2 [4] if to each pair of distinct points x, y of X, there exists a pair of disjoint b-open sets, one containing x and the other containing y.

3. b-$\mathcal{I}$-T_0 Spaces

Definition 4. An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0 if for any distinct pair of points in X, there is a b-$\mathcal{I}$-open set containing one of the points but not the other.

Theorem 5. *An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0 if and only if for each pair of distinct points x, y of X, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$.*

Proof. Let $(X, \tau, \mathcal{I})$ be a b-$\mathcal{I}$-T_0 space and x, y be any two distinct points of X. There exists a b-$\mathcal{I}$-open set G containing x or y, say, x but not y. Then $X \backslash G$ is a b-$\mathcal{I}$-closed set which does not contain x but contains y. Since $\in b\mathcal{I}\,\mathrm{Cl}(\{y\})$ is the smallest b-$\mathcal{I}$-closed set containing y, $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset X \backslash G$, and so $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Consequently, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Conversely, let x, $y \in X$, $x \neq y$ and $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Then there exists a point $z \in X$ such that z belongs to one of the two sets, say, $b\mathcal{I}\,\mathrm{Cl}(\{x\})$ but not to $b\mathcal{I}\,\mathrm{Cl}(\{y\})$. If we suppose that $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$, then $z \in b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset b\mathcal{I}\,\mathrm{Cl}(\{y\})$, which is a contradiction. So $x \in X \backslash b\mathcal{I}\,\mathrm{Cl}(\{y\})$, where $X \backslash b\mathcal{I}\,\mathrm{Cl}(\{y\})$ is a b-$\mathcal{I}$-open set and does not contain y. This shows that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0. □

Definition 6. [7] Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Then $(X_0, \tau_{|X_0}, \mathcal{I}_{|X_0})$ is an ideal topological space with an ideal $\mathcal{I}_{|X_0} = \{I \in \mathcal{I} | I \subset X_0\} = \{I \cap X_0 | I \in \mathcal{I}\}$.

Lemma 7. *[6] Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $A \in B\mathcal{I}O(X)$ and X_0 is open in $(X, \tau, \mathcal{I})$, then $A \cap X_0 \in B\mathcal{I}O(X_0)$.*

Theorem 8. *Every open subspace of a b-$\mathcal{I}$-T_0 space is b-$\mathcal{I}$-T_0.*

Proof. Let Y be an open subspace of a b-$\mathcal{I}$-T_0 space $(X, \tau, \mathcal{I})$ and x, y be two distinct points of Y. Then there exists a b-$\mathcal{I}$-open set A in X containing x or y, say, x but not y. Now by Lemma 7, $A \cap Y$ is a b-$\mathcal{I}$-open set in Y containing x but not y. Hence $(Y, \tau_{|Y}, \mathcal{I}_{|Y})$ is b-$\mathcal{I}_{|Y}$-T_0. □

Definition 9. A function f: $(X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be point b-$\mathcal{I}$-closure one-to-one if and only if x, $y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, then $b\mathcal{I}\,\mathrm{Cl}(\{f(x)\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{f(y)\})$.

Theorem 10. *If f: $(X,\tau,\mathcal{I}) \to (Y,\sigma)$ is point-b-$\mathcal{I}$-closure one-to-one and $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_0, then f is one-to-one.*

Proof. Let x and y be any two distinct points of X. Since $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_0, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$ by Theorem 5. But f is point-b-$\mathcal{I}$-closure one-to-one implies that $b\mathcal{I}\,\mathrm{Cl}(\{f(x)\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{f(y)\})$. Hence $f(x) \neq f(y)$. Thus, f is one-to-one. □

Theorem 11. *Let f: $(X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a function from b-$\mathcal{I}$-T_0 space $(X,\tau,\mathcal{I})$ into a topological space (Y,σ). Then f is point-b-$\mathcal{I}$-closure one-to-one if and only if f is one-to-one.*

Proof. The proof follows from Theorem 10. □

Theorem 12. *Let f: $(X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{I})$ be an injective b-$\mathcal{I}$-irresolute function. If Y is b-$\mathcal{I}$-T_0, then $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_0.*

Proof. Let x, $y \in X$ with $x \neq y$. Since f is injective and Y is b-$\mathcal{I}$-T_0, there exists a b-$\mathcal{I}$-open set V_x in Y such that $f(x) \in V_x$ and $f(y) \notin V_x$ or there exists a b-$\mathcal{I}$-open set V_y in Y such that $f(y) \in V_y$ and $f(x) \notin V_y$ with $f(x) \neq f(y)$. By b-$\mathcal{I}$-irresoluteness of f, $f^{-1}(V_x)$ is b-$\mathcal{I}$-open set in $(X,\tau,\mathcal{I})$ such that $x \in f^{-1}(V_x)$ and $y \notin f^{-1}(V_x)$ or $f^{-1}(V_y)$ is b-$\mathcal{I}$-open set in $(X,\tau,\mathcal{I})$ such that $y \in f^{-1}(V_y)$ and $x \notin f^{-1}(V_y)$. This shows that $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_0. □

4. b-$\mathcal{I}$-T_1 Spaces

Definition 13. An ideal topological space $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_1 if to each pair of distinct points x, y of X, there exists a pair of b-$\mathcal{I}$-open sets, one containing x but not y and the other containing y but not x.

Theorem 14. *For an ideal topological space $(X,\tau,\mathcal{I})$, each of the following statements are equivalent:*

1. *$(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-T_1;*
2. *Each one point set is b-$\mathcal{I}$-closed in X;*
3. *Each subset of X is the intersection of all b-$\mathcal{I}$-open sets containing it;*

4. *The intersection of all b-$\mathcal{I}$-open sets containing the point $x \in X$ is the set $\{x\}$.*

Proof. (1)⇒(2): Let $x \in X$. Then by (1), for any $y \in X$, $y \neq x$, there exists a b-$\mathcal{I}$-open set V_y containing y but not x. Hence $y \in V_y \subset X \backslash \{x\}$. Now varying y over $X\backslash\{x\}$ we get $X\backslash\{x\} = \cup \{V_y\colon y \in X\backslash\{x\}\}$. So $X\backslash\{x\}$ being a union of b-$\mathcal{I}$-open set. Accordingly $\{x\}$ is b-$\mathcal{I}$-closed.

(2)⇒(1): Let x, $y \in X$ and $x \neq y$. Then by (2), $\{x\}$ and $\{y\}$ are b-$\mathcal{I}$-closed sets. Hence $X\backslash\{x\}$ is a b-$\mathcal{I}$-open set containing y but not x and $X\backslash\{y\}$ is a b-$\mathcal{I}$-open set containing x but not y. Therefore, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1.

(2)⇒(3): If $A \subset X$, then for each point $y \notin A$, there exists a set $X\backslash\{y\}$ such that $A \subset X\backslash\{y\}$ and each of these sets $X\backslash\{y\}$ is b-$\mathcal{I}$-open. Hence $A = \cap$ $\{ X\backslash\{y\}$: y $\in X\backslash A\}$ so that the intersection of all b-$\mathcal{I}$-open sets containing A is the set A itself.

(3)⇒(4): Obvious.

(4)⇒(1): Let x, $y \in X$ and $x \neq y$. Hence there exists a b-$\mathcal{I}$-open set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, there exists a b-$\mathcal{I}$-open set U_y such that $y \in U_y$ and $x \notin U_y$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1. □

Theorem 15. *Every open subspace of a b-$\mathcal{I}$-T_1 space is b-$\mathcal{I}$-T_1.*

Proof. Let A be an open subspace of a b-$\mathcal{I}$-T_1 space $(X, \tau, \mathcal{I})$. Let $x \in A$. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1, $X\backslash\{x\}$ is b-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$. Now, A being open, $A \cap (X\backslash\{\text{x}\}) = A\backslash\{x\}$ is b-$\mathcal{I}$-open in A by Lemma 7. Consequently, $\{x\}$ is b-$\mathcal{I}$-closed in A. Hence by Theorem 14, A is b-$\mathcal{I}$-T_1. □

Theorem 16. *Let X be a T_1 space and $f\colon (X, \tau) \to (Y, \sigma, \mathcal{I})$ a b-$\mathcal{I}$-closed surjective function. Then $(Y, \sigma, \mathcal{I})$ is b-$\mathcal{I}$-T_1.*

Proof. Suppose $y \in Y$. Since f is surjective, there exists a point $x \in X$ such that $y = f(x)$. Since X is T_1, $\{x\}$ is closed in X. Again by hypothesis, $f(\{x\}) = \{y\}$ is b-$\mathcal{I}$-closed in Y. Hence by Theorem 14, Y is b-$\mathcal{I}$-T_1. □

Definition 17. A point $x \in X$ is said to be a b-$\mathcal{I}$-limit point of A if and only if for each $V \in B\mathcal{I}O(X)$, $U \cap (\text{A} \setminus \{x\}) \neq \emptyset$ and the set of all b-$\mathcal{I}$-limit points of A is called the b-$\mathcal{I}$-derived set of A and is denoted by b-$\mathcal{I}$d(A).

Theorem 18. *If $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1 and $x \in$ b-$\mathcal{I}d(A)$ for some $A \subset X$, then every b-$\mathcal{I}$-neighbourhood of x contains infinitely many points of A.*

Proof. Suppose U is a b-$\mathcal{I}$-neighbourhood of x such that $U \cap A$ is finite. Let $U \cap A = \{x_1, x_2, \ldots. x_n\} = B$. Clearly B is a b-$\mathcal{I}$-closed set. Hence $V = (U \cap A)\backslash(B\backslash\{x\})$ is a b-$\mathcal{I}$-neighbourhood of point x and $V \cap (A\backslash\{x\}) = \emptyset$, which implies that $x \in b$-$\mathcal{I}$d(A), which contradicts our assumption. Therefore, the given statement in the theorem is true. □

Theorem 19. *In a b-$\mathcal{I}$-T_1 space $(X, \tau, \mathcal{I})$, b-$\mathcal{I}d(A)$ is b-$\mathcal{I}$-closed for any subset A of X.*

Proof. As the proof of the theorem is easy, it is omitted. □

Theorem 20. *Let f: $(X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ be an injective and b-$\mathcal{I}$-irresolute function. If $(Y, \sigma, \mathcal{I})$ is b-$\mathcal{I}$-T_1, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1.*

Proof. Proof is similar to Theorem 12 □

Definition 21. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-R_0 [3] if and only if for every b-$\mathcal{I}$-open sets contains the b-$\mathcal{I}$-closure of each of its singletons.

Theorem 22. *An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1 if and only if it is b-$\mathcal{I}$-T_0 and b-$\mathcal{I}$-R_0.*

Proof. Let $(X, \tau, \mathcal{I})$ be a b-$\mathcal{I}$-T_1 space. Then by definition and as every b-$\mathcal{I}$-T_1 space is b-$\mathcal{I}$-R_0, it is clear that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0 and b-$\mathcal{I}$-R_0 space. Conversely, suppose that $(X, \tau, \mathcal{I})$ is both b-$\mathcal{I}$-T_0 and b-$\mathcal{I}$-R_0. Now, we show that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1 space. Let $x, y \in X$ be any pair of distinct points. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0, there exists a b-$\mathcal{I}$-open set G such that $x \in G$ and $y \notin G$ or there exists a b-$\mathcal{I}$-open set H such that $y \in H$ and $x \notin H$. Suppose $x \in G$ and $y \notin G$. As $x \in G$ implies the $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset G$. As $y \notin G$, $y \notin b\mathcal{I}\,\mathrm{Cl}(\{x\})$. Hence $y \in H = X\backslash b\mathcal{I}\,\mathrm{Cl}(\{x\})$ and it is clear that $x \notin H$. Hence, it follows that there exist b-$\mathcal{I}$-open sets G and H containing x and y respectively such that $y \notin G$ and $x \notin H$. This implies that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1. □

5. b-$\mathcal{I}$-T_2 Spaces

Definition 23. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-T_2 space if for each pair of distinct points x, y of X, there exists a pair of disjoint b-$\mathcal{I}$-open sets, one containing x and the other containing y.

Theorem 24. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following statements are equivalent:*

1. *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2;*
2. *Let $x \in X$. For each $y \neq x$, there exists $U \in B\mathcal{I}O(X, x)$ and $y \in b\mathcal{I}\,\mathrm{Cl}(U)$.*
3. *For each $x \in X$, $\cap\{b\mathcal{I}\,\mathrm{Cl}(U_x) : U_x$ is a b-$\mathcal{I}$-neighbourhood of x$\} = \{x\}$.*
4. *The diagonal $\triangle = \{(x, x) : x \in X\}$ is b-$\mathcal{I}$-closed in $X \times X$.*

Proof. (1)⇒(2): Let $x \in X$ and $y \neq x$. Then there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$. Clearly, $X \backslash V$ is b-$\mathcal{I}$-closed, $b\mathcal{I}\,\mathrm{Cl}(U) \subset X \backslash V$ and therefore $y \notin b\mathcal{I}\,\mathrm{Cl}(U)$.

(2)⇒(3): If $y \neq x$, then there exists $U \in B\mathcal{I}O(X, x)$ and $y \notin b\mathcal{I}\,\mathrm{Cl}(U)$. So $y \notin \cap\{b\mathcal{I}\,\mathrm{Cl}(U) : U \in B\mathcal{I}O(X, x)\}$.

(3)⇒(4): We prove that $X \backslash \triangle$ is b-$\mathcal{I}$-open. Let $(x, y) \notin \triangle$. Then $y \neq x$ and since $\cap\{b\mathcal{I}\,\mathrm{Cl}(U) : U \in B\mathcal{I}O(X, x)\} = \{x\}$, there is some $U \in B\mathcal{I}O(X, x)$ and $y \notin b\mathcal{I}\,\mathrm{Cl}(U)$. Since $U \cap X \backslash b\mathcal{I}\,\mathrm{Cl}(U) = \emptyset$, $U \times (X \backslash b\mathcal{I}\,\mathrm{Cl}(U))$ is b-$\mathcal{I}$-open set such that $(x, y) \in U \times (X \backslash b\mathcal{I}\,\mathrm{Cl}(U)) \subset X \backslash \triangle$.

(4)⇒(5): If $y \neq x$, then $(x, y) \notin \triangle$ and thus there exist $U, V \in B\mathcal{I}O(X)$ such that $(x, y) \in U \times V$ and $(U \times V) \cap \triangle = \emptyset$. Clearly, for the b-$\mathcal{I}$-open sets U and V we have $x \in U$, $y \in V$ and $U \cap V = \emptyset$. □

Corollary 25. *An ideal toological space is $(X, \tau, \mathcal{I})$ b-$\mathcal{I}$-T_2 if and only if each singleton subsets of X is b-$\mathcal{I}$-closed.*

Corollary 26. *An ideal toological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2 if and only if two distinct points of X have disjoint b-$\mathcal{I}$-closure.*

Theorem 27. *Every b-$\mathcal{I}$-regular T_0-space is b-$\mathcal{I}$-T_2.*

Proof. Let $(X, \tau, \mathcal{I})$ be a b-$\mathcal{I}$-regular T_0 space and x, $y \in X$ such that $x \neq y$. Since X is T_0, there exists an open set V containing one of the points, say, x but not y. Then $y \in X \backslash V$, $X \backslash V$ is closed and $x \notin X \backslash V$. By b-$\mathcal{I}$-regularity of X, there exist b-$\mathcal{I}$-open sets G and H such that $x \in G$, $y \in X \backslash V \subset H$ and $G \cap H = \emptyset$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2. □

Theorem 28. *Every open subspace of a b-$\mathcal{I}$-T_2 space is b-$\mathcal{I}$-T_2.*

Proof. Proof is similar to Theorem 15 □

Theorem 29. *If $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is injective, open and b-$\mathcal{I}$-continuous and Y is T_2, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2.*

Proof. Since f is injective, $f(x) \neq f(y)$ for each x, $y \in X$ and $x \neq y$. Now Y being T_2, there exist open sets G, H in Y such that $f(x) \in G$, $f(y) \in H$ and $G \cap H = \emptyset$. Let $U = f^{-1}(G)$ and $V = f^{-1}(H)$. Then by hypothesis, U and V are b-$\mathcal{I}$-open in X. Also $x \in f^{-1}(G) = U$, $y \in f^{-1}(H) = V$ and $U \cap V = f^{-1}(G) \cap f^{-1}(H) = \emptyset$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2. □

Definition 30. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called strongly b-$\mathcal{I}$-open if the image of every b-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$ is b-$\mathcal{J}$-oen in $(Y, \sigma, \mathcal{J})$.

Theorem 31. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, R an equivalence relation in X and $p : (X, \tau, \mathcal{I}) \to X|R$ the identification function. If $R \subset (X \times X)$ and p is a strongly b-$\mathcal{I}$-open function, then $X|R$ is b-$\mathcal{I}$-T_2*

Proof. Let $p(x)$ and $p(y)$ be the distinct members of $X|R$. Since x and y are not related, $R \subset (X \times X)$ is b-$\mathcal{I}$-closed in $X \times X$. There are b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$ and $U \times V \subset X \backslash R$. Thus $p(U)$ and $p(V)$ are disjoint b-$\mathcal{I}$-open sets in $X|R$ since p is strongly b-$\mathcal{I}$-open. □

Definition 32. [3] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-R_1 if for x, y in X with $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, there exists disjoint b-$\mathcal{I}$-open sets U and V such that $b\mathcal{I}\,\mathrm{Cl}(\{x\})$ is a subset of U and $b\mathcal{I}\,\mathrm{Cl}(\{y\})$ is a subset of V.

Theorem 33. *The ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2 if and only if it is b-$\mathcal{I}$-R_1 and b-$\mathcal{I}$-T_0.*

Proof. The proof is similar to Theorem 22 and thus omitted. □

Remark 34. In the following diagram we denote by arrows the implications between the seperation axioms which we have introduced and discussed in this paper and examples show that no other implications hold between them.

$$\begin{array}{ccccc}
T_2 & \Rightarrow & b\text{-}\mathcal{I}\text{-}T_2 & \Rightarrow & b\text{-}T_2 \\
\Downarrow & & \Downarrow & & \Downarrow \\
T_1 & \Rightarrow & b\text{-}\mathcal{I}\text{-}T_1 & \Rightarrow & b\text{-}T_2 \\
\Downarrow & & \Downarrow & & \Downarrow \\
T_0 & \Rightarrow & b\text{-}\mathcal{I}\text{-}T_0 & \Rightarrow & b\text{-}T_0
\end{array}$$

Example 35. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_i ($i = 0, 1, 2$) but not T_i ($i = 0, 1, 2$).

Example 36. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0 but not b-$\mathcal{I}$-T_1.

Example 37. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}$ $\{c\}, \{b, c\}\}$. Then $(X, \tau, \mathcal{I})$ is b-T_i (i = 0, 1, 2) but not b-$\mathcal{I}$-T_i (i = 0, 1, 2).

Theorem 38. *(1). An ideal topological space $(X, \tau, \{\emptyset\})$ is b-$\mathcal{I}$-T_0 (resp. b-$\mathcal{I}$-T_1, b-$\mathcal{I}$-T_2) if and only if it is b-T_0 (resp. b-T_1, b-T_2).*

(2). An ideal topological space $(X, \tau, \mathcal{N})$ is b-$\mathcal{I}$-T_0 (resp. b-$\mathcal{I}$-T_1, b-$\mathcal{I}$-T_2) if and only if it is b-T_0 (resp. b-T_1, b-T_2) ($\mathcal{N}$ is the ideal of all nowhere dense sets of X).

(3). An ideal topological space $(X, \tau, \mathcal{P}(X))$ is b-$\mathcal{I}$-T_0 (resp. b-$\mathcal{I}$-T_1, b-$\mathcal{I}$-T_2) if and only if it is T_0 (resp. T_1, T_2).

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