

**SMALL PERTURBATIONS AND INFINITESIMAL
DEFORMATIONS ON SURFACES OF REVOLUTION**

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Abstract: Any given surface of revolution embedded in Euclidean three-space can always be perturbed by arbitrarily small ambient isotopies as to admit highly nontrivial vector fields inducing infinitesimal deformations. For this matter Morse Theory is used, clarifying and giving a generalization of a problem originally introduced by M. Spivak [10] in a modern perspective.

AMS Subject Classification: 53A05, 53C05

Key Words: manifold, surface of revolution, perturbation, bending, infinitesimal deformation, Morse theory

1. Introduction

There are many different ways in which a geometric structure may be rigid, unwarpable, bendable or deformable. A precise formulation of any of these intuitions may be, for example, macroscopic or infinitesimal and may involve the notion of an (ambient) isotopy or not. The interest in this type of problem

Received: March 25, 2014

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matured in the late part of the nineteenth century (see [11]) and since then, the vast accumulated literature on the subject has also covered all sorts of applications in Architecture, Engineering and Mechanics (see [4], [5], [6], [8]). In any event, the paper of Stefan Cohn-Vossen, published in 1930, (see [2]) is certainly a classic approach to our main subject.

From the mathematical point of view, the partial differential equations and the analytic methods involved have been a matter of recent interest (see [12] and [13]). Despite the wealth of important results in the area, Shing-Tung Yau states, for example, that “the study of the rigidity of nonconvex surfaces is still in its infancy” (see [12, p. 27]).

The main result of this work is:

Let $\lambda = (r, h) : S^1 \rightarrow \mathbb{R}^2$ be an embedding such that $r(s) > 0$, $\forall s \in S^1$, and let $\mathcal{K} = \lambda(S^1)$. Then, for any compact set $B \subseteq \mathbb{R}^2$ containing \mathcal{K} in its interior, there is an “admissible” perturbation $\alpha : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ supported in B such that the surface of revolution obtained by revolving the perturbation of λ at time one by α admits a nontrivial infinitesimal deformation.

This theorem starts with an idea of M. Spivak (see [10, Vol. 5, pp. 253-260]) about the infinitesimal deformations of surfaces of revolution. Spivak works with a particular kind of surface diffeomorphic to the two dimensional sphere satisfying a restrictive analytical condition at its poles. In contrast, our methods work for general surfaces of revolution, diffeomorphic either to the 2-sphere or to the 2-dimensional torus. For this matter, the elementary part of Morse Theory is used, thus introducing a new way of combining analytical and geometric methods which allows substantial generalization and clarification. This is done in section 3, where further remarks are given.

2. Infinitesimal Rigidity

Definition. Let N be a manifold and let $\varphi : N \rightarrow \mathbb{R}^m$ be an embedding. A *bending* of φ is a smooth map $\alpha : [0, 1] \times N \rightarrow \mathbb{R}^m$ such that

- (a) each map $\bar{\alpha}(t) : N \rightarrow \mathbb{R}^m$, given by $p \mapsto \alpha(t, p)$, is an embedding,
- (b) $\bar{\alpha}(0) = \varphi$
- (c) for all $t \in [0, 1]$, the pull-back $\bar{\alpha}(t)^*(\langle, \rangle)$ equals $\varphi^*(\langle, \rangle)$.

The bending $\alpha : [0, 1] \times N \rightarrow \mathbb{R}^m$ is called *trivial* if each $\bar{\alpha}(t)$ is $A_t \circ \varphi$ for some isometry A_t of the ambient space \mathbb{R}^m ; it is called *nontrivial* if at least one

$\bar{\alpha}(t)$ is not of this form. We say that the embedding φ is *bendable* if there is a nontrivial bending of φ ; otherwise it is called *unbendable*.

N be a manifold and let TN denote its tangent bundle. For $G : N \rightarrow \mathbb{R}^m$, let $TG : TN \rightarrow T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ be its derivative and let $dG : TN \rightarrow \mathbb{R}^m$ be the projection of TG on the second coordinate. Let $U \subseteq \mathbb{R}^k$ be open and let $G : U \rightarrow \mathbb{R}^m$ be smooth. Denote by $JG : U \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ the Jacobian matrix of G . Endow \mathbb{R}^m with the Riemannian metric defined by its dot product \bullet . If A is a matrix, denote its transpose by A^t .

Definition. Let $\varphi : N \rightarrow \mathbb{R}^m$ be an embedding. A vector field $Z : N \rightarrow \mathbb{R}^m$ is an infinitesimal deformation for φ if it satisfies:

$$dZ(X) \bullet d\varphi(X) = 0 \quad \forall X \in \mathfrak{X}(N)$$

Infinitesimal deformations arising from trivial bendings are called *trivial*. If φ admits only trivial infinitesimal deformations, then it is called *infinitesimally rigid*.

It is not difficult to verify the following proposition.

Proposition. Let N be a manifold of dimension k and let $\varphi : N \rightarrow \mathbb{R}^m$ be an embedding. A vector field $Z : N \rightarrow \mathbb{R}^m$ is an infinitesimal deformation for φ if and only if there is an atlas $\mathcal{A} = \{(U, \beta) \mid \beta : U \rightarrow \mathbb{R}^k\}$ of N such that, for each $\beta \in \mathcal{A}$, the matrix $[J(Z \circ \beta^{-1})]^t \cdot J(\varphi \circ \beta^{-1})$ is antisymmetric.

3. Perturbations of Surfaces of Revolution

Let S^1 be the unit circle thought of as the real numbers modulo 2π (ie. $s \sim (\cos s, \sin s), \forall s \in \mathbb{R}$). Therefore we may think of a circle point as a unit complex number or as a lateral class $s + 2\pi \cdot \mathbb{Z}$. Let $\lambda = (r, h) : S^1 \rightarrow \mathbb{R}^2$ be an embedding such that $r(s) > 0, \forall s \in S^1$. Set $\gamma(t) = (\cos t, \sin t, 0)$ and $\hat{k} = (0, 0, 1)$. Let $F : S^1 \times \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $F(s, t) = r(s) \cdot \gamma(t) + h(s) \cdot \hat{k}$. Therefore, $\mathcal{S} = F(S^1 \times \mathbb{R})$ is a smooth surface and F is a local embedding. With some ambiguity we make the following definition. **Definition.** $\mathcal{F}(\lambda) = F$ (or \mathcal{S}) is

called the *surface of revolution obtained by revolving the curve λ around the vertical axis and (globally) parametrized by F* .

Definitions. Let N be a topological space and set $l = 0$; or let N be a smooth manifold and set $l = 3$. A C^l map $\alpha : [0, 1] \times N \rightarrow N$ is an *isotopy* of N if (a) each map $\bar{\alpha}(t) : N \rightarrow N$, given by $p \mapsto \alpha(t, p)$, is a C^l homeomorphism, (b) $\bar{\alpha}(0) = Id : N \rightarrow N$.

Let $\alpha : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an isotopy of \mathbb{R}^2 and let $\mathcal{K} \subseteq \{(x, z) \in \mathbb{R}^2 \mid x > 0\}$. We say that α is an *admissible perturbation for \mathcal{K}* if $\bar{\alpha}(t)(\mathcal{K}) \subseteq \{(x, z) \in \mathbb{R}^2 \mid x > 0\}$ for all $t \in [0, 1]$.

Let $G : N \rightarrow N$ be a C^l map. The closure of the set $\{p \in N \mid G(p) \neq p\}$ is the *support* of G and is denoted by $\text{supp}(G)$. For an isotopy α define its *support* as the closure of the set $\bigcup \{\text{supp } \bar{\alpha}(t) \mid t \in [0, 1]\}$.

The next statement is the main result of this work.

Theorem. *Let $\lambda = (r, h) : S^1 \rightarrow \mathbb{R}^2$ be an embedding such that $r(s) > 0$, $\forall s \in S^1$, and let $\mathcal{K} = \lambda(S^1)$. Then, for any compact set $B \subseteq \mathbb{R}^2$ containing \mathcal{K} in its interior, there is an admissible perturbation α for \mathcal{K} supported in B such that $\mathcal{F}(\bar{\alpha}(1) \circ \lambda)$ admits a nontrivial infinitesimal deformation.*

Definition. Let N be a manifold and let $g : N \rightarrow \mathbb{R}$ be smooth. A point $p \in N$ is *critical* if $\text{dg}|_p = 0$. Let $\mathcal{C}(g)$ be the set of critical points of g in N . A critical point p is *nondegenerate* if, for some (hence any) coordinate chart β around p , the Hessian matrix of $g \circ \beta^{-1}$ at $\beta(p)$ is nonsingular. Functions whose critical points are nondegenerate are called *Morse functions*.

For an arbitrary oriented one dimensional manifold $\mathcal{K} \subseteq \mathbb{R}^2$, let $\kappa : \mathcal{K} \rightarrow \mathbb{R}$ denote its (signed) curvature and let $\zeta : \mathcal{K} \rightarrow \mathbb{R}$ be the restriction to \mathcal{K} of the projection $(x, z) \mapsto z$.

Lemma 1. *Let $\mathcal{K} \subseteq \mathbb{R}^2$ be a compact one dimensional manifold. Suppose the height function $\zeta : \mathcal{K} \rightarrow \mathbb{R}$ is Morse. Then*

(a) $\mathcal{C}(\zeta)$ is finite,

(b) $\kappa(p) \neq 0$ for all $p \in \mathcal{C}(\zeta)$.

(c) For each $p \in \mathcal{C}(\zeta)$, there is a coordinate chart $\beta : U \rightarrow \langle -\epsilon, \epsilon \rangle$ around $p = \beta^{-1}(0)$ such that $\zeta \circ \beta^{-1}(u) = \zeta(p) \pm u^2$, for all $u \in \langle -\epsilon, \epsilon \rangle$.

Proof. For any Morse function the set of critical points is closed and discrete (see Hirsch [3, p. 143]), hence $\mathcal{C}(\zeta)$ is finite, for \mathcal{K} is compact.

Let $p \in \mathcal{C}(\zeta)$. By Morse's Lemma ([3, p. 145]), there is a coordinate chart $\beta : U \rightarrow \langle -\epsilon, \epsilon \rangle$ around $p = \beta^{-1}(0)$ such that $z(u) = \zeta(p) + \sigma \cdot u^2$, for all $u \in \langle -\epsilon, \epsilon \rangle$, where $z = \zeta \circ \beta^{-1}$ and $\sigma = 1$ if the index of p is zero and $\sigma = -1$ if the index of p is one. Let $x : \langle -\epsilon, \epsilon \rangle \rightarrow \mathbb{R}$ be the first coordinate of β^{-1} , so $\beta^{-1} = (x, z)$. Since $z'(0) = 0$, and β is a chart, $x'(0) \neq 0$. Let

$\phi : \langle -\epsilon, \epsilon \rangle \rightarrow \langle -\epsilon, \epsilon \rangle$ be such that $\phi(u) = -u$. If β is nonoriented, substitute β for $\beta \circ \phi$, so we may always assume that β is oriented if necessary. In any case, z is unchanged. Now,

$$\kappa(p) = \frac{\det((\beta^{-1})'(0), (\beta^{-1})''(0))}{\|(\beta^{-1})'(0)\|^3} = \frac{x'(0) \cdot z''(0)}{\|(\beta^{-1})'(0)\|^3} = \frac{x'(0) \cdot 2\sigma}{\|(\beta^{-1})'(0)\|^3}$$

showing that $\kappa(p) \neq 0$ □

Assertion. Let $\mathcal{K} \subseteq \mathbb{R}^2$ be a compact one dimensional manifold. For each $\nu \in S^1$ let $f_\nu : \mathcal{K} \rightarrow \mathbb{R}$ be the map $f_\nu(p) = \nu \bullet p$, so the orthogonal projection into the line through ν is given by $p \mapsto f_\nu(p) \cdot \nu$. Then the set of $\nu \in S^1$ such that f_ν is a Morse function is open and dense.

Proof. This is a minor instance of Hirsch [3, Ex. 2, p. 148] □

Corollary. Let $\mathcal{K} \subseteq \mathbb{R}^2$ be a compact one dimensional manifold. Let R_θ denote the plane rotation at angle θ in the positive direction with center at the origin and let $\mathcal{K}_\theta = R_\theta(\mathcal{K})$. Then, for almost every $\theta \in \mathbb{R}$, the height function $\zeta_\theta : \mathcal{K}_\theta \rightarrow \mathbb{R}$ is Morse.

Proof. Let $\hat{\nu} = (0, 1)$; the correspondence $\theta \mapsto R_{-\theta}(\hat{\nu})$ is a continuous bijection from $[0, 2\pi \rangle$ onto S^1 ; let $\hat{\nu} = R_\theta(\nu)$. Then, $f_\nu = \zeta_\theta \circ R_\theta$ for $\zeta_\theta(R_\theta(p)) = \hat{\nu} \bullet R_\theta(p) = R_\theta(\nu) \bullet R_\theta(p) = \nu \bullet p = f_\nu(p)$. Since $R_\theta : \mathcal{K} \rightarrow \mathcal{K}_\theta$ is a diffeomorphism, then f_ν is Morse iff ζ_θ is □

Definition. Let $\mathcal{K} \subseteq \mathbb{R}^2$ be a one dimensional manifold with Morse height function ζ and let $p \in \mathcal{C}(\zeta)$. Let x be the horizontal coordinate $(x, z) \mapsto x$. Then x must be a diffeomorphism around p ; therefore, $\zeta \circ x^{-1}$ is locally defined. The point p is called *even analytic* if $\zeta \circ x^{-1}$ is a real even analytic function around $x(p)$. \mathcal{K} is *even analytic (at $\mathcal{C}(\zeta)$)* if all its (nondegenerate) critical points are even analytic.

Lemma 2. $\mathcal{K} \subseteq \{(x, z) \in \mathbb{R}^2 \mid x > 0\}$ be a compact one dimensional manifold. Then, for any compact set $B \subseteq \mathbb{R}^2$ containing \mathcal{K} in its interior, there is an admissible perturbation α for \mathcal{K} supported in B and arbitrarily close to the identity such that $\bar{\alpha}(1)(\mathcal{K})$ has Morse height function ζ_1 and is even analytic (at $\mathcal{C}(\zeta_1)$).

Proof. Let $B \subseteq \mathbb{R}^2$ be compact containing \mathcal{K} in its interior B° . Assume, without loss of generality that $B \subseteq \{(x, z) \in \mathbb{R}^2 \mid x > 0\}$. Let $C \subseteq \mathbb{R}^2$ be

such that $\mathcal{K} \subseteq C^\circ \subseteq C \subseteq B^\circ$. Let $\eta : \mathbb{R}^2 \rightarrow [0, 1]$ be a smooth (bump) function with $\eta|_C = 1$ and $\eta|_{\mathbb{R}^2 \setminus B^\circ} = 0$. Let X be the vector plane field given by $X(x, z) = \eta(x, z) \cdot (-z, x), \forall (x, z) \in \mathbb{R}^2$. The one parameter group of diffeomorphisms $R : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by X is such that every $\bar{R}(t)$ is supported in B and there exists $\epsilon > 0$ where $R(t, p)$ is the rotation at angle t in the positive direction with center at the origin ($\forall |t| < \epsilon$ and $\forall p \in \mathcal{K}$).

Choose $0 < \theta < \epsilon$ such that the height function $\zeta_\theta : \bar{R}(\theta)(\mathcal{K}) \rightarrow \mathbb{R}$ is Morse. Note that θ may be chosen as small as desired (ϵ is not necessarily small).

Let $p = (x_0, z_0) \in \mathcal{C}(\zeta_\theta)$ be a concave critical point. So choose $\delta_1 > 0$ such that

- (1) $\rho = \zeta_\theta \circ x^{-1}$ is defined on $\langle x_0 - 2\delta_1, x_0 + 2\delta_1 \rangle$,
- (2) ρ' is strictly decreasing on $\langle x_0 - \delta_1, x_0 + \delta_1 \rangle$.

By making δ_1 smaller, elementary Taylor theory says that there exist $K < 0$ such that

- (3) $\rho(x) \leq z_0 + K \cdot (x - x_0)^2$ for all $x \in \langle x_0 - \delta_1, x_0 + \delta_1 \rangle$.

By making δ_1 smaller, now we can find $\delta_2 > 0$ such that

- (4) $[x_0 - \delta_1, x_0 + \delta_1] \times [z_0 - \delta_2, z_0 + 3\delta_2/2] \subseteq B^\circ$,
- (5) $z_0 - \delta_2/2 < \rho(x)$ for all $x \in \langle x_0 - \delta_1, x_0 + \delta_1 \rangle$,
- (6) $(\langle x_0 - \delta_1, x_0 + \delta_1 \rangle \times \langle z_0 - \delta_2, z_0 + 3\delta_2/2 \rangle) \cap \bar{R}(\theta)(\mathcal{K})$ is equal to $\{(x, \rho(x)) \mid x \in \langle x_0 - \delta_1, x_0 + \delta_1 \rangle\}$.

Let $\eta_1 : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

- (7) $\eta_1 = 1$ on $[x_0 - \delta_1/2, x_0 + \delta_1/2]$,
- (8) $\eta_1 = 0$ out of $\langle x_0 - \delta_1, x_0 + \delta_1 \rangle$,
- (9) η_1 is increasing on $\langle x_0 - \delta_1, x_0 - \delta_1/2 \rangle$ and decreasing on $\langle x_0 + \delta_1/2, x_0 + \delta_1 \rangle$.

Let $\eta_2 : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

- (10) $\eta_2 = 1$ on $[z_0 - \delta_2/2, z_0 + \delta_2]$,
- (11) $\eta_2 = 0$ out of $\langle z_0 - \delta_2, z_0 + 3\delta_2/2 \rangle$.

Define $f(x) = z_0 + K(x - x_0)^2$, which is certainly even analytic at x_0 , and let Y be the vector field given by $Y(x, z) = (0, \eta_2(z)\eta_1(x)(f(x) - \rho(x))), \forall (x, z) \in \mathbb{R}^2$, (let $Y(x, z) = (0, 0)$ when $\rho(x)$ fails to exist). Note that Y is vertical and points upwards when is different than zero. The one parameter group of

diffeomorphisms $S : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by Y is such that every $\bar{S}(t)$ is supported in $[x_0 - \delta_1, x_0 + \delta_1] \times [z_0 - \delta_2, z_0 + 3\delta_2/2]$. The flow for Y in the box $[0, 1] \times \mathbb{R} \times \langle z_0 - \delta_2/2, z_0 + \delta_2 \rangle$ is given by $(t, (x, z)) \mapsto (x, z + t\eta_1(x)(f(x) - \rho(x)))$. For $(t, x) \in [0, 1] \times \langle x_0 - \delta_1, x_0 + \delta_1 \rangle$, (and condition (5)), we have $(t, (x, \rho(x))) \mapsto (x, \rho(x) + t\eta_1(x)(f(x) - \rho(x)) = (x, g_t(x))$. In time one, for $x \in \langle x_0 - \delta_1/2, x_0 + \delta_1/2 \rangle$, we have $g_1(x) = f(x)$. Now, $g'_t(x) = (1 - \eta_1(x)) \rho'(x) + \eta_1(x) f'(x) + \eta'_1(x) (f(x) - \rho(x))$. By (2),(3),(7),(8) and (9) above $g'_t(x)$ has x_0 as its only zero. Therefore, by (6), the perturbation $\bar{R}(\theta)(\mathcal{K}) \rightarrow \bar{S}(t)(\bar{R}(\theta)(\mathcal{K}))$ does not introduce new critical points and $\bar{S}(1)(\bar{R}(\theta)(\mathcal{K}))$ is even analytic at p .

The situation for convex critical points is treated in exactly the same way. Hence, we can perturb $\bar{R}(\theta)(\mathcal{K})$ at all points in $\mathcal{C}(\zeta_\theta)$ (disjointly) in order to make all resulting critical points even analytic □

Lemma 2 and the following Proposition imply the Theorem of this section.

Proposition. *Let $\mathcal{K} \subseteq \{(x, z) \in \mathbb{R}^2 \mid x > 0\}$ be diffeomorphic to the unit circle S^1 . Suppose \mathcal{K} has Morse height function ζ which is also even analytic (at $\mathcal{C}(\zeta)$). Let $\lambda = (r, h) : S^1 \rightarrow \mathcal{K}$ be a diffeomorphism, so the surface of revolution $\mathcal{F}(\lambda)$ is defined. Then, for any compact set $B \subseteq \mathbb{R}^2$ containing \mathcal{K} in its interior, there is an admissible perturbation α for \mathcal{K} supported in B such that $\mathcal{F}(\bar{\alpha}(1) \circ \lambda)$ admits nontrivial infinitesimal deformations different from zero almost everywhere.*

Proof. (cf Spivak [10, Vol. 5, pp. 253-260]). Let $F = \mathcal{F}(\lambda)$. Then, any vector field Z along F can be written uniquely as

$$Z(s, t) = a(s, t) \cdot \gamma(t) + b(s, t) \cdot \gamma'(t) + c(s, t) \cdot \hat{k}$$

for some smooth functions a, b, c . An infinitesimal deformation Z for F is trivial if it is of the form $Z(s, t) = \mu \times F(s, t) + \omega$, where $\mu, \omega \in \mathbb{R}^3$ and \times denotes the cross product.

Since F is a global regular parametrization, its local inverses form an atlas for its image. Hence, by the Proposition of section 2, Z is an infinitesimal deformation if and only if $(JZ)^t \cdot JF$ is a 2×2 antisymmetric matrix. That is,

$$\frac{\partial Z}{\partial s} \bullet \frac{\partial F}{\partial s} = 0, \quad \frac{\partial Z}{\partial t} \bullet \frac{\partial F}{\partial t} = 0, \quad \frac{\partial Z}{\partial t} \bullet \frac{\partial F}{\partial s} + \frac{\partial Z}{\partial s} \bullet \frac{\partial F}{\partial t} = 0 .$$

A simple calculation shows that these equations are explicitly given by

$$r' \frac{\partial a}{\partial s} + h' \frac{\partial c}{\partial s} = 0, \quad \frac{\partial b}{\partial t} + a = 0, \quad r' \left(\frac{\partial a}{\partial t} - b \right) + r \frac{\partial b}{\partial s} + h' \frac{\partial c}{\partial t} = 0 \quad (\dagger)$$

Following Spivak, we propose solutions of the form

$$a(s, t) = \sum_{k=-\infty}^{\infty} e^{ikt} \phi_k(s), b(s, t) = \sum_{k=-\infty}^{\infty} e^{ikt} \psi_k(s), c(s, t) = \sum_{k=-\infty}^{\infty} e^{ikt} \xi_k(s)$$

For a (complex-valued) solution involving a single k , equations (†) become

$$\left. \begin{aligned} r' \phi'_k + h' \xi'_k &= 0 \\ ik \psi_k + \phi_k &= 0 \\ r'(ik \phi_k - \psi_k) + r\psi'_k + ik h' \xi_k &= 0 \end{aligned} \right\} \quad (\dagger\dagger)$$

If this system is solved, then it is easy to see that $a_k(s, t) = e^{ikt} \phi_k(s) + e^{-ikt} \bar{\phi}_k(s)$, $b_k(s, t) = e^{ikt} \psi_k(s) + e^{-ikt} \bar{\psi}_k(s)$, $c_k(s, t) = e^{ikt} \xi_k(s) + e^{-ikt} \bar{\xi}_k(s)$ are the scalar components of a real infinitesimal deformation.

Differentiating the third equation in (††) and using the other two, we obtain the equivalent system

$$\left. \begin{aligned} \phi_k + ik \psi_k &= 0 \\ ik h' \xi_k + (k^2 - 1)r' \psi_k + r\psi'_k &= 0 \\ r\psi''_k + (k^2 - 1)r'' \psi_k + ik h'' \xi_k &= 0 \end{aligned} \right\} \quad (\dagger \dagger \dagger)$$

System († † †) is easily transformed into system

$$\left. \begin{aligned} \phi_k + ik \psi_k &= 0 \\ ik h' \xi_k + (k^2 - 1)r' \psi_k + r\psi'_k &= 0 \\ r \left[\frac{\psi'_k}{h'} \right]' + (k^2 - 1) \left[\frac{r'}{h'} \right]' \psi_k &= 0 \end{aligned} \right\} \quad (\dagger \dagger \dagger \dagger)$$

Solutions of the third equation above immediately give solutions of the first two. Hence we can define infinitesimal deformations away from the critical set $\{F(s, t) \mid h'(s) = 0\}$. Now, ζ is given by $(x, z) \mapsto z$ and if we let x to be the horizontal coordinate $(x, z) \mapsto x$, then $r = x \circ \lambda$ and $h = \zeta \circ \lambda$. Also, \mathbb{R} is (periodically) partitioned into

- (a) open intervals where h' is nonzero and,
- (b) its endpoints, which correspond to the elements in $\mathcal{C}(\zeta) \subseteq \mathbb{R}^2$.

We shall refer to the closures of these intervals (or its images under h or λ) as “the segments”. In particular, for any $s \in \mathbb{R}$, if $h'(s) = 0$ then r is locally invertible around s . For such points let $\rho = h \circ r^{-1} = \zeta \circ x^{-1}$, so ρ is even analytic. Incidentally, this proves that the number of elements in $\mathcal{C}(\zeta)$, say $\# | \mathcal{C}(\zeta) |$, is even, because intervals where h' is positive and negative must alternate, for $\lambda^{-1}(\mathcal{C}(\zeta))$ contains no inflection points. Clearly $2 \leq \# | \mathcal{C}(\zeta) |$, for the top and bottom points in \mathcal{K} are critical. Also, the equality $h \circ r^{-1} = \zeta \circ x^{-1}$ shows that if $s_{\flat} = s_{\natural} + 2 \cdot \pi$ correspond to the same point in $\mathcal{C}(\zeta)$ then the associated ρ is the same. Observe that $\{s \in \mathbb{R} \mid r'(s) = 0\}$ may be rather nasty.

Define, when meaningful, $\tilde{\Phi}_k = \phi_k \circ h^{-1}$, $\tilde{\Psi}_k = \psi_k \circ h^{-1}$, $\tilde{\Xi}_k = \xi_k \circ h^{-1}$ and $\Phi_k = \phi_k \circ r^{-1}$, $\Psi_k = \psi_k \circ r^{-1}$, $\Xi_k = \xi_k \circ r^{-1}$, which in turn satisfy the respective systems

$$\left. \begin{aligned} \tilde{\Phi}_k + ik \tilde{\Psi}_k &= 0 \\ ik \tilde{\Xi}_k + (r \circ h^{-1})\tilde{\Psi}'_k + (k^2 - 1)(r \circ h^{-1})'\tilde{\Psi}_k &= 0 \\ (r \circ h^{-1})\tilde{\Psi}''_k + (k^2 - 1)(r \circ h^{-1})''\tilde{\Psi}_k &= 0 \end{aligned} \right\} \quad (5 - \dagger)$$

and $\Phi_k + ik \Psi_k = 0$, $ik \rho' \Xi_k + \text{Id } \Psi'_k + (k^2 - 1)\Psi_k = 0$, $\text{Id } \rho' \Psi''_k - \text{Id } \rho'' \Psi'_k - (k^2 - 1)\rho'' \Psi_k = 0$. Finally, the second equation of this last system may be changed to obtain the equivalent system

$$\left. \begin{aligned} \Phi_k + ik \Psi_k &= 0 \\ ik \Xi_k + \left(\frac{\rho'}{\rho''}\right) \text{Id } \Psi''_k &= 0 \\ \text{Id } \rho' \Psi''_k - \text{Id } \rho'' \Psi'_k - (k^2 - 1)\rho'' \Psi_k &= 0 \end{aligned} \right\} \quad (6 - \dagger)$$

Note that, when both r^{-1} and h^{-1} are defined, then $\tilde{\Phi}_k = \Phi_k \circ \rho^{-1}$, $\tilde{\Psi}_k = \Psi_k \circ \rho^{-1}$ and $\tilde{\Xi}_k = \Xi_k \circ \rho^{-1}$. Again, solutions for the third equation in (5 - †) immediately produce solutions for the first two away from critical points. In the same manner, solutions for the third equation in (6 - †) immediately produce solutions for the first two around critical points.

Fix $p \in \mathcal{C}(\zeta)$. Then, ρ is (even) analytic around $x_0 = x(p)$ with zero derivative at x_0 . Then it can be expressed as $\rho(x) = \zeta(p) + (1/2!)\rho''(x_0)(x - x_0)^2 + (1/4!)\rho''''(x - x_0)^4 + \dots$, with $\rho''(x_0) \neq 0$. Hence we can write $\rho'(x) = (x - x_0)P(x)$ with $P(x_0) = \rho''(x_0)$. Multiplying the third equation in (6 - †) by $(x - x_0)/xP(x)$ we get

$$(x - x_0)^2 \Psi''_k(x) + (x - x_0) \left(-\frac{\rho''(x)}{P(x)}\right) \Psi'_k(x) + \left(- (k^2 - 1) \frac{(x - x_0)\rho''(x)}{xP(x)}\right) \Psi_k(x) = 0$$

This is a standard expression of a second-order differential equation having a regular singular point at x_0 with roots 0 and 2 for its “indicial equation”. Therefore, it has a unique analytic solution of the form $\Psi_k(x) = (x - x_0)^2 A(x)$, satisfying $A(x_0) = 1$, (see Birkhoff and Rota [1, p. 252 ff]). In particular, x_0 is an isolated zero for $\Psi_k(x)$. Note that $x_0 > 0$, but this is a geometric condition that assures that we have a smooth surface of revolution. From the analytic point of view, $x_0 = 0$ is admissible. In this case the indicial equation has roots $1 \pm k$. If $k > 1$, there is exactly one analytic solution (around 0) of the form $\Psi_k(x) = x^{1+k} A(x)$, (cf [10, Vol. 5, p. 257]).

Let $s_1 < s_2 < s_3$ be three consecutive points in $\lambda^{-1}(\mathcal{C}(\zeta))$, where $s_3 = s_1 + 2 \cdot \pi$ is possible. Then there is an $\epsilon > 0$ such that the system $(\dagger\dagger\dagger)$ has well determined nontrivial smooth solutions ϕ_k, ψ_k, ξ_k in the interval $\langle s_1 - \epsilon, s_2 \rangle$. In particular, $\psi_k(s_1) = \psi'_k(s_1) = 0$ and $\xi_k(s_1) = 0$.

It is true that if $\psi_k(s_2)$ is defined and equal to zero, then the solutions ϕ_k, ψ_k, ξ_k are all well defined in the interval $\langle s_1 - \epsilon, s_2 + \epsilon \rangle$ and therefore, in the interval $\langle s_1 - \epsilon, s_3 \rangle$. The obstruction to this continuation process is that, usually, we expect to have $\psi_k(s_2) = \pm\infty$.

Up to now k has been kept fixed. To solve the above mentioned obstruction we will have to vary this integer and perturb \mathcal{K} (a third time, but now away from $\mathcal{C}(\zeta)$).

So let $s_1 < s_2$ be two consecutive points in $\lambda^{-1}(\mathcal{C}(\zeta))$. In $[s_1, s_2]$ the function h is monotone increasing or decreasing. Let $\{h(s_1), h(s_2)\} = \{z_1, z_2\}$ be labeled so that $z_1 < z_2$. Let $R = r \circ h^{-1} : [z_1, z_2] \rightarrow \langle 0, \infty \rangle$ be the horizontal coordinate for this segment. Fix a point $z_0 \in \langle z_1, z_2 \rangle$ and let $x_0 = R(z_0)$. Choose $\delta_1, \delta_2 > 0$ such that

- (1) $z_1 < z_0 - 3\delta_2 < z_0 + 3\delta_2 < z_2$,
- (2) $[x_0 - 2\delta_1, x_0 + 2\delta_1] \times [z_0 - 3\delta_2, z_0 + 3\delta_2] \subseteq B^\circ$,
- (3) $0 < x_0 - \delta_1/2 < R(z) < x_0 + \delta_1/2, \forall z \in \langle z_0 - 3\delta_2, z_0 + 3\delta_2 \rangle$,
- (4) $[x_0 - 2\delta_1, x_0 + 2\delta_1] \times [z_0 - 3\delta_2, z_0 + 3\delta_2] \cap \mathcal{K} = \{(R(z), z) \mid z \in [z_0 - 3\delta_2, z_0 + 3\delta_2]\}$.

As in Lemma 2, we can perturb the graph of R on $[z_0 - 3\delta_2, z_0 + 3\delta_2]$ within the box $[x_0 - \delta_1/2, x_0 + \delta_1/2] \times [z_0 - 5\delta_2/2, z_0 + 5\delta_2/2]$, by a small flow T induced by a horizontal vector field, in such a way that $[x_0 - \delta_1/2, x_0 + \delta_1/2] \times [z_0 - 2\delta_2, z_0 + 2\delta_2] \cap T(1)(\mathcal{K})$ is the graph of a smooth strictly concave function $f_0 : [z_0 - 2\delta_2, z_0 + 2\delta_2] \rightarrow \langle x_0 - \delta_1/2, x_0 + \delta_1/2 \rangle$. It is clear that this perturbation

does not introduce new critical points in $\bar{T}(1)(\mathcal{K})$. Let $\tilde{R}_0 : [z_1, z_2] \rightarrow \langle 0, \infty \rangle$ denote the (new) horizontal coordinate for the segment $[z_1, z_2]$.

Let $f_1 : [z_0 - 2\delta_2, z_0 + 2\delta_2] \rightarrow \langle x_0 - \delta_1/2, x_0 + \delta_1/2 \rangle$ be a smooth function such that

(5) f_0 and f_1 have the same values and the same first and second derivatives at $z_0 \pm 2\delta_2$,

(6) f_1 is strictly concave in $[z_0 - 2\delta_2, z_0 - \delta_2] \cup \langle z_0 + \delta_2, z_0 + 2\delta_2 \rangle$,

(7) f_1 is strictly convex in $\langle z_0 - \delta_2, z_0 + \delta_2 \rangle$,

(8) $f_1(z) \leq f_0(z), \forall z \in [z_0 - 2\delta_2, z_0 + 2\delta_2]$.

The horizontal field $X(x, z) = (f_1(z) - f_0(z), 0), \forall z \in [z_0 - 2\delta_2, z_0 + 2\delta_2]$ and $X(x, z) = (0, 0), \forall z \notin [z_0 - 2\delta_2, z_0 + 2\delta_2]$ is C^2 and may be damped by a bump function (in variable x) in order to produce a small flow that sends, in time one, the graph of f_0 onto the graph of f_1 . Let $\{\tilde{R}_t : [z_1, z_2] \rightarrow \langle 0, \infty \rangle\}_{t \in [0,1]}$ be the continuous family of horizontal coordinates produced by the flow (for the segment $[z_1, z_2]$). Fix $k \in \mathbb{N}$ and consider the family of equations (for $t \in [0, 1]$, on $\langle z_1, z_2 \rangle$)

$$\tilde{\Psi}'' + \left[\frac{(k^2 - 1)(\tilde{R}_t)''}{\tilde{R}_t} \right] \tilde{\Psi} = 0 \quad (\clubsuit)$$

and let $\{\tilde{\Psi}_{k,t}\}_{t \in [0,1]}$ be a continuous family of solutions such that its expression in the horizontal axis has $R(z_1)$ as a zero of order two. Then,

(9) For all $t \in [0, 1]$, $\tilde{R}_t = \tilde{R}_0$ in $[z_1, z_0 - 2\delta_2]$. Hence, for any $t_a, t_b \in [0, 1]$, $\tilde{\Psi}_{k,t_a}|_{[z_1, z_0 - 2\delta_2]} = \tilde{\Psi}_{k,t_b}|_{[z_1, z_0 - 2\delta_2]}$

(10) For all $t \in [0, 1]$, $\tilde{R}_t = \tilde{R}_0$ in $[z_0 + 2\delta_2, z_2]$ so $\{\tilde{\Psi}_{k,t}\}_{t \in [0,1]}$ is a family of solutions of the same differential equation in that interval. Therefore, by the Sturm Comparison Theorem, for any $t_a, t_b \in [0, 1]$, $-1 \leq \#[(\tilde{\Psi}_{k,t_a})^{-1}(\{0\})] - \#[(\tilde{\Psi}_{k,t_b})^{-1}(\{0\})] \leq 1$, where, as before, $\#[V]$ denotes the cardinality of set V .

(11) By concavity of $f_0 = \tilde{R}_0$ in $[z_0 - 2\delta_2, z_0 + 2\delta_2]$, $\tilde{\Psi}_{k,0}$ has at most one zero in the interval.

(12) By concavity of $f_1 = \tilde{R}_1$ in $[z_0 - 2\delta_2, z_0 - \delta_2] \cup \langle z_0 + \delta_2, z_0 + 2\delta_2 \rangle$, $\tilde{\Psi}_{k,1}$ has at most one zero in each of these intervals.

Considerations (9 - 12) hold for any k . Now, if k is sufficiently large, by the Sturm Comparison Theorem and the convexity of \tilde{R}_1 in $\langle z_0 - \delta_2, z_0 + \delta_2 \rangle$, $\tilde{\Psi}_{k,1}$ can be made to have many zeroes in the interval. For such value of k , $\#[(\tilde{\Psi}_{k,0})^{-1}(\{0\})] < \#[(\tilde{\Psi}_{k,1})^{-1}(\{0\})]$.

Let $\hat{t} \in [0, 1]$ be a point of discontinuity of $t \mapsto \#[(\tilde{\Psi}_{k,t})^{-1}(\{0\})]$. We want to prove that $\tilde{\Psi}_{k,\hat{t}}(z_2) = 0$. Let $\{\Upsilon_{0,t}, \Upsilon_{\infty,t}\}_{t \in [0,1]}$ be a family of linearly independent solutions of equation (♣) in $\langle z_1, z_2 \rangle$ such that $\Upsilon_{0,t}(z_2) = 0$ and $\Upsilon_{\infty,t}(z_2) \neq 0$ (possibly $\pm\infty$), for all $t \in [0, 1]$. Then there exist unique functions $d_0, d_\infty : [0, 1] \rightarrow \mathbb{R}$ satisfying $\tilde{\Psi}_{k,t} = d_0(t) \cdot \Upsilon_{0,t} + d_\infty(t) \cdot \Upsilon_{\infty,t}$. Since the Wronskian at z_0 is not zero, then

$$\begin{bmatrix} d_0(t) \\ d_\infty(t) \end{bmatrix} = \begin{bmatrix} \Upsilon_{0,t}(z_0) & \Upsilon_{\infty,t}(z_0) \\ \Upsilon'_{0,t}(z_0) & \Upsilon'_{\infty,t}(z_0) \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\Psi}_{k,t}(z_0) \\ \tilde{\Psi}'_{k,t}(z_0) \end{bmatrix}$$

This relation, and the C^1 uniform continuity in compact sets, of the state space and parameter space, of the solutions of differential equations implies that the coefficients d_0, d_∞ are continuous.

Suppose that $\tilde{\Psi}_{k,\hat{t}}(z_2) \neq 0$. Then $d_\infty(\hat{t}) \neq 0$ and certainly $\Upsilon_{0,\hat{t}}(z_2) = 0$. This and the continuity of the situation imply that there exist $\nu_1, \nu_2 > 0$ and a neighbourhood $U \subseteq [0, 1]$ of \hat{t} such that $\nu_1 < |\tilde{\Psi}_{k,t}(z)|$ for all $t \in U$ and all $z \in [z_2 - 2\nu_2, z_2 >$.

In particular, all the zeroes of $\tilde{\Psi}_{k,\hat{t}}$ are in $[z_1, z_2 - \nu_2]$. Since it is a non-trivial solution of a second order differential equation, it is true that $(\forall z)(z \in \langle z_1, z_2 - \nu_2 \rangle \wedge \tilde{\Psi}_{k,\hat{t}}(z) = 0 \Rightarrow \tilde{\Psi}'_{k,\hat{t}}(z) \neq 0)$. This implies that, for $|t - \hat{t}|$ sufficiently small, all functions $\tilde{\Psi}_{k,t}$ have the same number of zeroes (see item (9) above); contradicting the assumption that \hat{t} was a point of discontinuity for $t \mapsto \#[(\tilde{\Psi}_{k,t})^{-1}(\{0\})]$. Hence $\tilde{\Psi}_{k,\hat{t}}(z_2) = 0$.

By applying the above constructions to all segments in a given order, the Proposition is proved □

Remarks. (1) It is not difficult to verify that, for $k \geq 2$, infinitesimal (nonzero) deformations obtained just by solving system († † †) are nontrivial. Also, the “Fourier” nature of the solutions proposed says that nonzero solutions for different k 's are linearly independent (see Spivak [10, Vol. 5, p. 256]).

(2) The “glueing” of function f_1 with f_0 in item (5) of the above Proposition may be much improved so as to make the subsequent arguments as smooth as pleased. But suppose $s_1 < s_2 < \dots < s_{\ell+1}$, where $s_1 = s_{\ell+1} \pmod{2\pi \mathbb{Z}}$, is a sequence of numbers corresponding to the points in $\mathcal{C}(\zeta)$. The continuation process through the various s_j 's can be done as smoothly as the solutions to system (♣) are; except at the end when one finally reaches point s_1 again. No matter how smooth are the original data to the problem, in general, the glued solutions at s_1 are to be only of class C^2 (if the family $\{\tilde{R}_t\}_{t \in [0,1]}$ is C^4).

(3) Any topological embedding $S^1 \hookrightarrow \mathbb{R}^2$ can certainly be perturbed by arbitrarily small isotopies resulting in a C^∞ embedding. Nevertheless, “tracking differentiability” may always prove to be of use in later situations (see Rado [9]).

(4) Let $(\mathcal{K}, \partial\mathcal{K}) \subseteq (\{(x, z) \in \mathbb{R}^2 \mid x \geq 0\}, \{0\} \times \mathbb{R})$ be diffeomorphic to the unit interval $([0, 1], \partial[0, 1])$ in such a way that \mathcal{K} intersects the vertical axis orthogonally. Then the surface of revolution obtained by revolving \mathcal{K} is topologically a sphere subject to the methods of this work. Indeed, first consider the one-manifold $\mathcal{K} \cup M(\mathcal{K})$, where $M : (x, z) \mapsto (-x, z)$. The analog of Lemma 2 can be obtained without difficulty as well as the succeeding Proposition. At the set of poles $\partial\mathcal{K}$ is where the fact that the height function is Morse even analytic, and not just Morse analytic, becomes handy. These are points of null abscissa hence the indicial equation has roots $1 \pm k$ and therefore, the best is to start the continuation process from there. Also, is easy to imagine examples of noncompact surfaces of revolution for which our main result is true.

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