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THE PARTIAL DIFFERENTIAL OPERATOR $\diamondsuit^k_{m,c}$ RELATED TO THE WAVE EQUATION AND LAPLACIAN

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Abstract: In this article, we study the elementary solution of the operator $\diamondsuit^k_{m,c}$,iterated k-times and is defined by

$$\diamondsuit^k_{m,c} = \left[\left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \frac{m^2}{2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} - \frac{m^2}{2}\right)^2\right]^k,$$

where $p+q=n, c, m$ are positive real number and n is the dimension of Euclidean space $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and k is a nonnegative integer. We obtain the elementary solution depending on the conditions of c, p, q, k and m.

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1. Introduction

Bupasiri [1] has first introduced the operator $\diamondsuit^k_{m,c}$ iterated k-times which is defined by

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$$\diamondsuit_{m,c}^{k} = \left[\left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}+\frac{m^2}{2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}-\frac{m^2}{2}\right)^2\right]^k$$
$$= \left[\left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}+m^2\right)\left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}+\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k,$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, k is a nonnegative integer. The operator $\diamondsuit_{m,c}^{k}$ related to the diamond operator and denote by

$$\diamondsuit_{0,1}^{k} = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1}$$

Thus, equation (**??**) can be written as

$$\diamondsuit_{m,c}^{k} = \triangle_c^k(\Box_c+m^2)^k = (\Box_c+m^2)^k\triangle_c^k, \tag{2}$$

where

$$\triangle_c^k = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}+\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k, \tag{3}$$

$$(\Box_c+m^2)^k = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}+m^2\right)^k \tag{4}$$

and

$$\Box_c^k = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \tag{5}$$

is the operator related to the ultra-hyperbolic operator.
The operator $(\Box_c+m^2)^k$ related to the ultra-hyperbolic Klein Gordon operator, iterated k-times, and is defined by

$$(\Box_1+m^2)^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}+m^2\right)^k, \tag{6}$$

where

$$\Box_1^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \tag{7}$$

is the ultra-hyperbolic operator, iterated k-times. By putting $p = 1, c = 1$ and $x_1 = t$ (time) in (7) then we obtain the wave operator

$$\Box_1 = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} \tag{8}$$

and from (**??**) with $q = 0, c = 1, m = 0$ and $k = 1$, we obtain

$$\diamondsuit_{0,1} = \triangle_1^2 \tag{9}$$

where

$$\triangle_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}. \tag{10}$$

Bupasiri [1] can find the elementary solution of the operator related to the ultra-hyperbolic Klein Gordon operator, iterated k-times, and is defined by

$$(\Box_c + m^2)^k = \left[\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \right]^k, \quad p + q = n. \tag{11}$$

We obtain the elementary solution $W_{2k,c}(u, m)$, defined by

$$W_{2k,c}(u, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k + r)}{r! \Gamma(k)} (m^2)^r R^H_{2k+2r,c}(u), \tag{12}$$

where $R^H_{2k+2r,c}(u)$ is defined by (14).

The purpose of this work, we can find the elementary solution of the operator $\diamondsuit^k_{m,c}$. Moreover, we found that the relationship between the elementary solution of the wave operator defined by (8) and the Laplace operator defined by (9) and (10).

2. Preliminary Notes

Definition 1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$u = c^2 \left(x_1^2 + x_2^2 + \cdots + x_p^2 \right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{13}$$

where $p+q=n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([4], p.72) that

$$R^H_{\alpha,c}(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{14}$$

$R^H_{\alpha,c}(x) = R^H_{\alpha,1}(x)$ is called the *ultra-hyperbolic kernel of Marcel Riesz*. Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)} \tag{15}$$

and p is the number of positive terms of

$$u = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p+q=n$$

and let supp $R^H_{\alpha,c}(x) \subset \overline{\Gamma}_+$. Now $R^H_{\alpha,c}(x)$ is an ordinary function if Re $(\alpha,c) \geq n$ and is a distribution of α if Re $(\alpha,c) < n$. Now, if $p=1$ then (14) reduces to the function $M_{\alpha,c}(u)$ say, and defined by

$$M_{\alpha,c}(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{16}$$

where $u = c^2x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1}\Gamma(\frac{\alpha-n+2}{2})$. The function $M_{\alpha,c}(u) = M_{\alpha,1}(u)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and

$$v = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2, \qquad p+q=n. \tag{17}$$

For any complex number β , we define the function

$$R^e_{\beta,c}(v) = 2^{-\beta}\pi^{-n/2}\Gamma\left(\frac{n-\beta}{2}\right)\frac{v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \tag{18}$$

The function $R^e_{\beta,c}(v)$ is called the *elliptic kernel of Marcel Riesz*. It is an ordinary function if $\text{Re}(\beta,c) \geq n$ and a distribution of β if $\text{Re}(\beta,c) < n$.

Lemma 3. *Given the equation $\triangle_c^k u(x) = \delta(x)$ for $x \in \mathbb{R}^n$, where $\triangle_c^k$ is the operator iterated k-times defined by (3). Then*

$$u(x) = (-1)^k R^e_{2k,c}(v)$$

where $R^e_{2k,c}(v)$ defined by (18) with $\beta = 2k$.

Proof. see [2]. □

Lemma 4. *Given the equation* $(\Box_c + m^2)^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}^n$, *where* $(\Box_c + m^2)^k$ *is the operator iterated* k*-times defined by (11) . Then*

$$u(x) = W_{2k,c}(u,m)$$

where $W_{2k,c}(u,m)$ *defined by (12).*

Proof. see [1]. □

Lemma 5. *The function* $W_{-2k,c}(u,m)$ *and* $(-1)^k R^e_{-2k,c}(v)$ *are the inverses in the convolution algebras of* $W_{2k,c}(u,m)$ *and* $(-1)^k R^e_{2k,c}(v)$ *respectively.*

Proof. We need to show that

$$W_{-2k,c}(u,m) * W_{2k,c}(u,m) = W_{-2k+2k,c}(u,m) = W_{0,c}(u,m) = \delta$$

and

$$(-1)^k R^e_{-2k,c}(v) * (-1)^k R^e_{2k,c}(v) = R^e_{-2k+2k,c}(v) = R^e_{0,c}(v) = \delta.$$

To prove these, see Bupasiri ([1], p. 6654-6655), Tellez ([5], p. 107-110), Tellez & Trione ([6], p. 123), Trione (1987, p.10) and Donoghue ([3], p. 118, p. 158). □

3. Main Results

Theorem 6. *Given the equation*

$$\diamondsuit^k_{m,c} G(x) = \delta(x), \tag{19}$$

where $\diamondsuit^k_{m,c}$ *is the operator iterated* k*-times defined by (**??**),* $\delta(x)$ *is the Dirac delta distribution,* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* k *is a nonnegative integer. Then we obtain*

$$G(x) = (-1)^k R^e_{2k,c}(v) * W_{2k,c}(u,m) \tag{20}$$

is elementary solution of the equation (19), where $R^e_{2k,c}(v)$ *and* $W_{2k,c}(u,m)$ *are defined by (18) and (12) respectively with* $\beta = 2k$*. Moreover, from (20)*

$$(-1)^k R^e_{-2k,c}(v) * G(x) = W_{2k,c}(u,m) \tag{21}$$

as elementary solution of the operator $(\Box_c + m^2)^k$ *iterated* k*-times defined by (11) and in particular from (20) and (21) with* $p = 1, q = n-1, c = 1, k = 1, x_1 = t$ *and* $m = 0$*, we obtain*

$$(-1)^k R^e_{-2,1}(v) * G(x) = M_{2,1}(u) \tag{22}$$

as elementary solution of the wave operator defined by (8) where $M_{2,1}(u)$ *is defined by (17) with* $\alpha = 2$*. Also, for* $q = 0, c = 1$ *and* $m = 0$ *then (19) become*

$$\triangle_1^{2k} G(x) = \delta(x) \tag{23}$$

and by (20) we obtain

$$G(x) = R^e_{4k,1}(v) \tag{24}$$

is elementary solution of (23) where $\triangle_1^{2k}$ *is the Laplacian of* p*-dimension, iterated* $2k$*-times and is defined by (10).*

Proof. From (19) and (2) we have

$$\diamondsuit^k_{m,c} G(x) = \left(\triangle_c^k (\Box_c + m^2)^k\right) G(x) = \delta(x) \tag{25}$$

convolving both side of the equation (25) by the $(-1)^k R^e_{2k,c}(v) * W_{2k,c}(u,m)$ and the properties of convolution with derivatives, we obtain

$$\triangle_c^k \left((-1)^k R^e_{2k,c}(v)\right) * (\Box_c + m^2)^k \left(W_{2k,c}(u,m)\right) * G(x) \\ = (-1)^k R^e_{2k,c}(v) * W_{2k,c}(u,m).$$

Thus $G(x) = \delta * \delta * G(x) = (-1)^k R^e_{2k,c}(v) * W_{2k,c}(u,m)$ by Lemma 3 and Lemma 4 We obtain (20) as require. Now from (20) and by Lemma 5 and the properties of inverses in the convolution algebra, we obtain

$$(-1)^k R^e_{-2k,c}(v) * G(x) = W_{2k,c}(u,m) \quad \text{(by Lemma 4)}$$

is elementary solution of the operator $(\Box_c + m^2)^k$ iterated k-times defined by (11).

In particular, by putting $p = 1, q = n-1, k = 1, c = 1, x_1 = t$ and $m = 0$ in (20) and (21), $W_{2,1}(u, m = 0) = R^H_{2,1}(u)$ reduces to $M_{2,1}(u)$ where $M_{2,1}(u)$ is defined by (17) with $\alpha = 2$. Thus we obtain

$$(-1)^k R^e_{-2,1}(v) * G(x) = M_{2,1}(u)$$

is elementary solution of the wave operator defined by (8) where $u = t^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2$. Also, for $q = 0, c = 1$ and $m = 0$ then (19) becomes

$$\triangle_1^{2k} G(x) = \delta \tag{26}$$

where $\triangle_1^{2k}$ is the Laplacian of p-dimension iterated $2k$-times. By Lemma 3, we have

$$G(x) = (-1)^{2k} R^e_{4k,1}(v) = R^e_{4k,1}(v)$$

is elementary solution of (26) where

$$v = x_1^2 + x_2^2 + \cdots + x_p^2.$$

Moreover, we can also find $G(x)$ from (20), since $q = 0, c = 1$ and $m = 0$, we have $W_{2k,1}(u, m = 0) = R^H_{2k,1}(u)$ reduces to $(-1)^k R^e_{2k,1}(v)$, where $v = x_1^2 + x_2^2 + \cdots + x_p^2$. Thus, by (20) for $q = 0, c = 1$ and $m = 0$, we obtain

$$\begin{aligned} G(x) &= (-1)^k R^e_{2k,1}(v) * (-1)^k R^e_{2k,1}(v) \\ &= (-1)^{2k} R^e_{2k+2k,1}(v) \\ &= R^e_{4k,1}(v) \quad \text{(by Donoghue [3])}. \end{aligned}$$

That complete the proofs. □

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