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# REFLEXIVE NON-DEROGATORY MATRICES

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**Abstract:** A matrix $A$ is called reflexive if and only if $Lat(A) \subseteq Lat(B)$ implies that $B = p(A)$ for some polynomial $p$. In this article, we characterize reflexive non-derogatory matrices.

**Key Words:** invariant subspace, reflexive matrix, Jordan block, non-derogatory matrix

## 1. Introduction

Let $M_n$ be the algebra all $n \times n$ complex matrices. For any matrix $A \in M_n$, a closed linear subspace $F$ of $\mathbb{C}^n$ is $A$-invariant if $A(F) \subseteq F$. We let $Lat(A)$ denote the lattice of all closed subspaces invariant for $A$, and $AlgLat(A)$ is the algebra of all matrices $B \in M_n$ such that $Lat(A) \subseteq Lat(B)$. A matrix $A \in M_n$ is said to be reflexive if $AlgLat(A) = W(A)$, where $W(A)$ is the weakly closed algebra generated by $A$ and $I$. Actually, $A \in M_n$ is reflexive if and only if $Lat(A) \subset Lat(B)$ implies that $B = p(A)$ for some polynomial $P$.

Let $A \in M_n$. If $m = \prod_{i=1}^{n} {p_i}^{r_i}$ is the minimal polynomial of $A$, with

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$p_1, p_2, ..., p_n$ distinct irreducible monic polynomials, we recall that the subspaces $M_i = ker p_i(A)^{r_i}$; $1 \leq i \leq n$, are invariant for $A$, are linearly independent, and span $\mathbb{C}^n$. Suppose that $A_i = A\mid_{M_i}$ for $i = 1, ..., n$. In [1] Deddens and Fillmore showed that if $A = \sum_{i=1}^{n} \bigoplus A_i$, then $A$ is reflexive if and only if each $A_i$ is reflexive. A square complex matrix $A \in M_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is similar to a block diagonal matrix

$$J = \begin{pmatrix} J_1(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_2) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & J_k(\lambda_k) \end{pmatrix},$$

where $1 \leq k \leq n$ and each block $J_i(\lambda_i)$ is a square matrix of the form

$$J_i(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & 0 & \lambda_i \end{pmatrix}.$$

So there exists an invertible matrix $P$ such that $P^{-1}AP = J$ is such that the only non-zero entries of $J$ are on the diagonal and the superdiagonal. The matrix $J$ is called the Jordan normal form of $A$. Each $J_k(\lambda)$ is called a Jordan block of $A$. A matrix $A$ is non-derogatory if and only if it's characteristic polynomial coincides with it's minimum polynomial. For some sources on these topics one can see [1–6].

## 2. Reflexivity of Non-Derogatory Matrices

For the proof of the main result we will need the following lemma.

**Lemma 2.1.** *Let $J_k(\lambda)$ be a $k \times k$ Jordan block. $Lat(J_k(\lambda))$ is*

$$E = \{0, span(e_1), span(e_1, e_2), \ldots, span(e_1, e_2, \ldots, e_{k-1}), \mathbb{C}^k\}.$$

*Proof.* It is easy to show that $E$ is $J_k(\lambda)$-invariant. It is sufficient that, if $W$ is a $J_k(\lambda)$-invariant subspace of $\mathbb{C}^k$ of dimension at least 1, then $W \in E$. Let $J_k(\lambda)W \subseteq W$, since $e_1$ is the only eigenvector of $J_k(\lambda)$, so $e_1 \in W$. Suppose

$e_2 \notin W$. For every $w \in W$, we have $w = (w_1, 0, w_2, \ldots, w_k)^T$. Hence $J_k(\lambda)W \subseteq W$ which implies that

$$\begin{pmatrix} \lambda & 1 & & 0 \\ 0 & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & 0 & \lambda \end{pmatrix} \begin{pmatrix} w_1 \\ 0 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} \lambda w_1 \\ w_3 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} w'_1 \\ 0 \\ \vdots \\ w'_k \end{pmatrix},$$

where $w' = \begin{pmatrix} w'_1 & 0 & w'_2 & \cdots & w'_k \end{pmatrix}^T \in W$. So $w_3 = 0$ and $e_3 \notin W$. It follows that if $e_i \notin W$ for $i = 2, 3, \ldots, k-1$, then $e_{i+1} \notin W$. Hence $W \in E$. □

**Theorem 2.2.** *Every Jordan block of degree at least 2 is not reflexive.*

*Proof.* Let $B = (B)_{i,j} \in M_n$ and $Lat(J_n(\lambda)) \subseteq Lat(B)$. So

$$E = \{0, span(e_1), span(e_1, e_2), \ldots, span(e_1, e_2, \ldots, e_{k-1}), \mathbb{C}^k\} \subseteq Lat(B).$$

Since $e_1 \in E$, we have

$$\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{n,1} \end{pmatrix},$$

which implies that $b_{2,1} = b_{3,1} = \cdots = b_{n,1} = 0$, and $\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \in E$. Hence

$$\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ 0 & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n,2} & \cdots & b_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_{1,1} + b_{1,2} \\ b_{2,2} \\ \vdots \\ b_{2,n} \end{pmatrix}$$

and so $b_{3,2} = b_{4,2} = \cdots = b_{n,2} = 0$. Consequently, $B$ should be an upper triangular matrix. In the other hand, $B = p(J_n(\lambda))$ implies that

$$B = c_0 I + c_1 J_n(\lambda) + \ldots + c_n (J_n(\lambda))^n,$$

where $c_i$ for $i = 1, 2, \ldots, n$ are constant complex numbers.

Hence

$$\begin{aligned} b_{11} &= c_0 + c_1\lambda + \ldots + c_n\lambda^n, \\ b_{22} &= c_0 + c_1\lambda + \ldots + c_n\lambda^n, \\ &\vdots \\ b_{nn} &= c_0 + c_1\lambda + \ldots + c_n\lambda^n. \end{aligned}$$

It follows that all entries on diagonal of $B$ are equal. So by the above discussion $J_n(\lambda)$ is not reflexive. □

**Corollary 2.3.** *Let $A$ be a non-derogatory matrix. Then $A$ is reflexive if and only if every eigenvalue of $A$ has algebraic multiplicity of 1.*

*Proof.* By using Theorem 1 in [1] and Theorem 2.2, the proof is complete. □

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