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REFLEXIVITY IN ASSOCIATIVE TRIPLE SYSTEMS

V. Ramaswamy[1], D. Rajalakshmi[2 §]

Principal, BIET
Davangere, Karnataka, 577 004, INDIA
[2]Department of Mathematics
Srinivasa Ramanujan Centre
SASTRA University, Kumbakonam, Tamil Nadu, 612 001, INDIA

Abstract: In the present paper, we define the notions of regularity, strong regularity and reflexivity in associative triple systems and prove some theorems concerning these notions.

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1. Introduction and Preliminaries

A vector space T over a field F together with a trilinear map $(x, y, z) \to < x, y, z >$ is called a triple system. A triple system T is said to be an associative triple system if $<< x, y, z >, u, v > = < x, < y, z, u >, v > = < x, y, < z, u, v >>$ for all $x, y, z, u, v \in T$. For example, if A is an associative algebra, then the underlying vector space of A together with the map $(x, y, z) \to xyz$ is an associative triple system which we denote by T_A. T_A is called the triple system associated with the algebra A. Hence forward, T will denote an associative triple system. We call an element x in T invertible if there exists an element $x_1 \in T$ such that $< x, x_1, t > = < x_1, x, t > = < t, x, x_1 > = < t, x_1, x > = t$ for all $t \in T$. x_1 is called an inverse of x. It can be easily seen that an invertible element of a triple system T has a unique inverse. Triple system T

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§Correspondence author

is said to be a division triple system if every non zero element of T is invertible. It can be easily shown that T is a division triple system if and only if the following
condition is satisfied: For $a,b,c \in T$ $(a,b \neq 0)$, the equations $< a,b,x > = c$, $< a,x,b > = c$ and $< x,a,b > = c$ have solutions. If A is a division algebra, then the triple system T_A associated with A is a division triple system.

The odd powers of an element $a \in T$ are defined recursively as follows. $a^1 = a$, $a^{[2(n+1)+1]} = < a^{(2n+1)}, a, a >$. a is said to be nilpotent if $a^{(2n+1)} = 0$ for some positive integer n. For $x \in T$, we define a map $U_x : T \to T$ as $U_x(y) = < x, y, x >$. x is said to be a zero divisor if U_x is not injective.

An element $a \in T$ is said to be regular if there exist an element $x \in T$ such that $a = U_a(x) = < a, x, a >$. x is referred to as a generalized inverse of a. a is said to be unit regular if there exists an invertible element x such that $a = U_a(x) = < a, x, a >$. a is said to be strongly regular if there exists an element $x \in T$ such that $a = < a, a, x >$. T is said to be strongly regular if every element of T is strongly regular. An element x is said to be a reflexive inverse of a if $a = U_a(x)$ and $x = U_x(a)$. If x is a reflexive inverse of a, then it follows that a is regular with x as a generalized inverse.

2. Main Results

The relationship between regularity and strong regularity is given by the following.

Theorem 2.1. For each element a of a strongly regular triple system T, there corresponds an element x such that $< a, a, x > = < a, x, a > = < x, a, a > = a$. Thus every element of a strongly regular triple system T is regular.

Proof. We will first prove that a strongly regular triple system T cannot contain any nonzero nilpotent element. Suppose $s \in T$ and s is nilpotent. Then $s^{(2p+1)} = 0$ for some positive integer p. Since T is strongly regular, there exists $t \in T$ such that $s = < s, s, t >$. Now $s = < s, s, t > = < s, < s, s, t >, t > = < s^3, t, t >$ (by associativity). We will now prove that $s = < s^{(2n+1)}, t^{(2n-1)}, t >$ by induction on n. The result is true when $n = 1$. Now assume that $s = < s^{(2n+1)}, t^{(2n-1)}, t >$. Then

$$\begin{aligned} s &=<< s^{(2n-1)}, s, s >, t^{(2n-1)}, t > \\ &=<< s^{(2n-1)}, s, < s, s, t >>, t^{(2n-1)}, t > \ \text{(note that } s =< s, s, t >) \\ &=< s^{(2n+1)}, s, t^{(2n+1)} > \ \text{(by associativity)} \end{aligned}$$

$$= < s^{(2n+1)}, < s, s, t >, t^{(2n+1)} > \text{ since } s = < s, s, t >$$
$$= < s^{(2n+3)}, t^{(2n+1)}, t > \text{ (by associativity)}$$

We have thus proved that $s = < s^{(2n+1)}, t^{(2n-1)}, t >$ for all positive integers n. Since $s^{(2p+1)} = 0$, we conclude that $s = 0$. We will now come to the proof of the theorem. Since T is strongly regular , so is a. Hence there exists an element x such that $a = < a, a, x >$. Using associativity of T ,we can easily prove that $(a- < a, x, a >) = 0$. Therefore $a- < a, x, a >$ is a nilpotent element of the strongly regular triple system T. By what we have proved above, we conclude that $a- < a, x, a > = 0$ showing that $a = < a, x, a >$. Similarly, we can prove that $a = < x, a, a >$. □

Theorem 2.2. For a nonzero regular element a of a triple system T, the following statements are equivalent:

(i) a has a unique generalized inverse.

(ii) a is not a zero divisor.

(iii) There exist an element $x \in T$ such that $< x, a, t > = < t, a, x > = t$ for all t.

Proof. (i) ⇒(ii) Let x be the unique generalized inverse of a. Then $a = U_a(x) = < a, x, a >$. Let $t \in T$ such that $U_a(t) = 0$. Now $U_a(x + t) = < a, x + t, a > = < a, x, a > + < a, t, a > = a$ since $< a, t, a > = 0$. Hence $x + t$ is also a generalized inverse of a. By the uniqueness of x , we conclude that $x + t = x$ so that $t = 0$. U_a is thus injective proving that a is not a zero divisor.

(ii)⇒(iii)

Since a is regular, we can find an element $x \in T$ such that $a = U_a(x) = < a, x, a >$. Let $t \in T$ be arbitrary. Then Since a is not a zero divisor, U_a is injective so that $t- < x, a, t > = 0$ proving that $t = < x, a, t >$. Similarly, we can prove that $t = < t, a, x >$ for all $t \in T$.

(iii)⇒(i)

Since a is regular, a has a generalized inverse say t so that $a = U_a(t) = < a, t, a >$. By (iii), there exists an element $x \in T$ such that $< x, a, t > = < t, a, x > = t$. Now $< x, a, x > = < x, < a, t, a >, x > = << x, a, t >, a, x > = < t, a, x > = t$. This proves that t is unique. □

Theorem 2.3. For a non zero element a of a strongly regular triple system T, the following statements are equivalent.

(i) a has a unique generalized inverse.

(ii) a is not a zero divisor.

(iii) a is invertible.

Proof. (i)⇒(ii) Since T is strongly regular, it follows from theorem that a is regular so that by theorem , a is not a zero divisor.

(ii)⇒(iii) Since T is strongly regular ,it follows from the theorem that there exists an element x such that $a = < a, a, x > = < a, x, a > = < x, a, a >$. Also by the theorem, $< x, a, t > = < t, a, x > = t$ for all $t \in T$. Now, $U_a(t- < a, x, t >) = < a, t- < a, x, t >, a > = < a, t, a > - < a, < a, x, t >, a > = < a, t, a > - << a, a, x >, t, a > = < a, t, a > - < a, t, a > = 0$. Since a is not a zero divisor, U_a is injective so that $t = < a, x, t >$. Similarly using $a = < x, a, a >$,we can prove that $t = < t, x, a >$. x is thus the inverse of a.

(iii) ⇒(i) Since by Theorem, a is regular, (iii) ⇒(i) follows from Theorem 2.2. □

Theorem 2.4. T is a division triple system if and only if every non zero element of T is strongly regular with a unique generalized inverse.

Proof. If T is a division triple system, then every nonzero element $a \in T$ has an inverse say $a^{(-1)}$. Hence $< a, a, a^{(-1)} > = a$ which proves that a is strongly regular. Thus T is a strongly regular triple system in which a is invertible so that by Theorem 2.3, a has a unique generalized inverse. Conversely, if every non zero element of T is strongly regular with a unique generalized inverse, then T is a strongly regular triple system so that by Theorem 2.3, every non zero element of T is invertible. Hence T is a division triple system. □

Lemma 2.5. If a is a regular element of T with a generalized inverse x, then $< x, a, x >$ is a reflexive inverse of a.

Proof. x is a generalized inverse of a means $a = U_a(x) = < a, x, a >$. If $y = < x, a, x >$, then $U_a(y) = < a, y, a > = < a, < x, a, x >, a > = << a, x, a >, x, a > = < a, x, a > = a$. Again $U_y(a) = < y, a, y > = << x, a, x >, a, < x, a, x >> = < x, < a, x, a >, < x, a, x >> = < x, a, < x, a, x >> = < x, < a, x, a >, x > = < x, a, x > = y$. This shows that y is a reflexive inverse of a. We call an element $x \in T$ a strong reflexive inverse of a if $< a, a, x > = < a, x, a > = < x, a, a > = a$ and $< x, x, a >= < x, a, x > = < a, x, x > = x$. □

Theorem 2.6. If a is an element of T with a unique reflexive inverse x, then $< a,a,x > = < x,a,a >$ and $< a,x,x > = < x,x,a >$.

Proof. Since x is a reflexive inverse of a , we have $U_a(x) = a$ and $U_x(a) = x$. For $y \in T$, we have $U_a(x + y- < x,a,y >) = < a,x + y- < x,a,y >,a >$ $= < a,x,a > + < a,y,a > - < a,< x,a,y >,a > = U_a(x) + U_a(y)-$ $<< a,x,a >,y,a > = U_a(x) + U_a(y) - U_a(y)$ since $< a,x,a > = U_a(x) = a$ $= U_a(x) = a$ Also $U_a(x + y- < y,a,x >) = < a,x + y- < y,a,x >,a >$ $= < a,x,a > + < a,y,a > - < a,< y,a,x >,a > = U_a(x) + U_a(y)-$ $< a,y,< a,x,a >> = U_a(x) + U_a(y) - U_a(y) = U_a(x) = a$. Thus $x + y-$ $< x,a,y >$ and $x + y- < y,a,x >$ are both generalized inverses of a. From the Lemma, it follows that $< x + y- < x,a,y >,a,x + y- < x,a,y >>$ and $< x + y- < y,a,x >,a,x + y- < y,a,x >>$ are both reflexive inverses of a. Since x is the unique reflexive inverse of a, we have $x = < x + y-$ $< x,a,y >,a,x + y- < x,a,y >>$. Using associativity of T, $< x,a,x >$ $= x$ and $< a,x,a > = a$ and simplifying, we obtain $x = x+ < y,a,x > -$ $<< x,a,y >,a,x >$ so that $< y,a,x > = << x,a,y >,a,x >$ for all $y \in T$. Similarly, using the fact that $< x + y- < y,a,x >,a,x + y- < y,a,x >>$ is a reflexive inverse of a, we can prove that $< x,a,y > = < x,a,< y,a,x >>$ for all $y \in T$. Thus for all $y \in T$,

$$< x,a,y >=< y,a,x > \tag{2.1}$$

Taking $y = a$ in 2.1, we obtain $< x,a,a > = < a,a,x >$. Again, taking y $= < x,x,a >$ in 2.1, we obtain $< x,a,< x,x,a >> = << x,x,a >,a,x >$ i.e $<< x,a,x >,x,a > = << x,x,a >,a,x >$. Since $< x,a,x > = x$, we have

$$< x,x,a >=<< x,x,a >,a,x >=< x,< x,a,a >,x > \tag{2.2}$$

Taking $y = < a,x,x >$ in 2.1, we obtain $< x,a,< a,x,x >> =$ $<< a,x,x >,a,x >$ so that $< x,< a,a,x >,x > = < a,x,< x,a,x >>$. i.e $< x,< a,a,x >,x > = < a,x,x >$ since $< x,a,x > = x$. Since we have already proved that $< a,a,x > = < x,a,a >$, this implies that

$$< x,< x,a,a >,x >=< a,x,x > \tag{2.3}$$

From 2.2 and 2.3 we obtain $< x,x,a > = < a,x,x >$. This completes the proof of the theorem. □

Theorem 2.7. Let a be an element of T with a reflexive inverse x such that $< x,x,a > = x$. Further, let x be the unique generalized inverse of a. Then a is invertible.

Proof. Since x is a reflexive inverse of a, we have $a = U_a(x) = < a, x, a >$ and $x = U_x(a) = < x, a, x >$. Since x is the unique generalized inverse of a , x is also the unique reflexive inverse of a. Hence it follows from Theorem 2.6 that $< a, a, x > = x, a, a >$ and $< a, x, x > = < x, x, a >$. This together with the hypothesis gives $< a, x, x > = < x, x, a > = < x, a, x > = x$. Now for any $t \in T$, we have $< a, x, t > = << a, x, a >, x, t > = < a, < x, a, x >, t >= < a, < x, x, a >, t >= << a, x, x >, a, t > = < x, a, t >$. Similarly, we can prove that $< t, a, x > = < t, x, a >$ for all $t \in T$. Now by Theorem 2.2, $< x, a, t > = < t, a, x >= t$ for all $t \in T$ so that $< a, x, t > = < x, a, t > = < t, a, x > = < t, x, a > = t$ for all $t \in T$ proving that a is invertible with inverse x. □

Theorem 2.8. Let a be an element of T possessing a reflexive inverse. Furthermore, let $< a, x, t > = < x, a, t >$ and $< t, a, x > = < t, x, a >$ for all $t \in T$ and for all generalized inverses x of a. Then a has a unique reflexive inverse.

Proof. Let y and z both be reflexive inverses of a. Then $a = < a, y, a > = < a, z, a >$, $y = < y, a, y >$ and $z = < z, a, z >$. Now $< a, y, t > = < y, a, t >$ (by hypothesis)$= < y, < a, z, a >, t > = << y, a, z >, a, t > = << a, y, z >, a, t >$ (by hypothesis since y is a generalized inverse)$= < a, y, < z, a, t >> = < a, y, < a, z, t >>$ (again by hypothesis since z is a generalized inverse) $= << a, y, a >, z, t > = < a, z, t >$. We have thus proved that $< a, y, t > = < y, a, t > = < a, z, t > = < z, a, t >$ for all $t \in T$. Similarly, we can prove that $< t, a, y > = < t, y, a > = < t, a, z > = < t, z, a >$ for all $t \in T$. Now $y = < y, a, y > = < z, a, y >$ (since by above, $< y, a, t > = < z, a, t >$ for all $t \in T$) $= < z, a, z >$ (since $< t, a, y > = < t, a, z >$ for all $t \in T$) $= z$. This proves that reflexive inverse of a is unique. □

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