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DIRICHLET-LAURICELLA TYPE D DISTRIBUTION

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Abstract: In this article, we define and study the Dirichlet-Lauricella type D distribution. This distribution is a generalization of the Dirichlet type 1 distribution.

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1. Introduction

The random variables $U_1, \ldots, U_n$ are said to have a Dirichlet type 1 distribution with parameters $a_1, \ldots, a_n; a_{n+1}$, denoted by $(U_1, \ldots, U_n) \sim \text{D1}(a_1, \ldots, a_n; a_{n+1})$, if their joint probability density function (p.d.f.) is given by

$$\frac{\Gamma(\sum_{i=1}^{n+1} a_i)}{\prod_{i=1}^{n+1} \Gamma(a_i)} \prod_{i=1}^{n} u_i^{a_i - 1} \left(1 - \sum_{i=1}^{n} u_i\right)^{a_{n+1} - 1},$$

$$u_i > 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} u_i < 1. \quad (1)$$

The Dirichlet type 1 distribution is a multivariate generalization of the beta distribution. Dirichlet distributions are very often used as prior distributions in Bayesian statistics, and in fact the Dirichlet distribution is the conjugate prior

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of the categorical distribution and multinomial distribution. The Dirichlet type 1 distribution has been studied extensively, for example, see Kotz, Balakrishnan and Johnson [2], and Gupta and Nagar [3].

In this article, we give a generalization of the Dirichlet type 1 distribution. This generalization is based on the Lauricella type D hypergeometric function and thus will be called Dirichlet-Lauricella type D distribution. Recently, Nagar and Gómez [8] have proposed a generalization of Dirichlet type 1 distribution based on the Lauricell's type B hypergeometric function.

In Section 2, we give definition of Lauricella type D hypergeometric function. We define Dirichlet-Lauricella type D distribution in Section 3. Section 4, deals with several properties such as marginal densities and joint moment.

2. Preliminaries

The Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$ and $(a)_0 = 1$. The Lauricella hypergeometric functions $F_D^{(n)}$ of several variables is defined as

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(a)_{j_1+\cdots+j_n}(b_1)_{j_1}\cdots(b_n)_{j_n}}{(c)_{j_1+\cdots+j_n}} \frac{z_1^{j_1}\cdots z_n^{j_n}}{j_1!\cdots j_n!}, \quad \max\{|z_1|, \ldots, |z_n|\} < 1. \tag{2}$$

For $n = 1$, the Lauricella hypergeometric functions $F_D^{(n)}$ reduces a Gauss hypergeometric function and for and $n = 2$ it slides to an Appell hypergeometric function F_1.

The integral representations of $F_D^{(n)}$ is given by

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_m) = \frac{\Gamma(c)}{\prod_{i=1}^n \Gamma(b_i)\Gamma(c - \sum_{i=1}^n b_i)} \times \int_{\substack{u_1>0,\ldots,u_n>0 \\ 1-\sum_{i=1}^n u_i>0}} \cdots \int \frac{\prod_{i=1}^n u_i^{b_i-1} (1-\sum_{i=1}^n u_i)^{c-\sum_{i=1}^n a_i - 1}}{(1-\sum_{i=1}^n z_i u_i)^a} \prod_{i=1}^n \mathrm{d}u_i, \tag{3}$$

where $\mathrm{Re}(b_i) > 0$, $i = 1, \ldots, n$ and $\mathrm{Re}(c - b_1 - \cdots - b_n) > 0$.

Another representation of $F_D^{(n)}$, in terms of a single integral, is given by

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{\prod_{i=1}^n (1-uz_i)^{b_i}} \mathrm{d}u,$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$ and $|\arg(1-z_i)| < \pi$, $i = 1, \ldots, n$.

For further results and properties of this function the reader is referred to Exton [1], Srivastava and Karlsson [11], and Prudnikov, Brychkov and Marichev [10, Sec. 7.2.4].

Let $f(\cdot)$ be a continuous function and $\alpha_i > 0$, $i = 1, \ldots, r$. The integral

$$D_r(\alpha_1, \ldots, \alpha_r; f) = \underset{\substack{x_1>0,\ldots,x_r>0\\ \sum_{i=1}^r x_i<1}}{\int\cdots\int} \prod_{i=1}^r x_i^{\alpha_i-1} f\left(\sum_{i=1}^r x_i\right) \prod_{i=1}^r \mathrm{d}x_i \tag{4}$$

is known as the Liouville-Dirichlet integral. Substituting $y_i = x_i/x, i = 1, \ldots, r-1$ and $x = \sum_{i=1}^r x_i$ with the Jacobian $J(x_1, \ldots, x_{r-1}, x_r \to y_1, \ldots, y_{r-1}, x) = x^{r-1}$ it is easy to see that

$$D_n(\alpha_1, \ldots, \alpha_r; f) = \frac{\prod_{i=1}^r \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^r \alpha_i)} \int_0^1 x^{\sum_{i=1}^r \alpha_i - 1} f\,(x)\, \mathrm{d}x. \tag{5}$$

Further, by taking $f(x) = (1-x)^{\alpha_{r+1}}$, $\alpha_{r+1} > 0$ in (4) and (5), the classical Dirichelet integral is evaluated as

$$\underset{\substack{x_1>0,\ldots,x_r>0\\ \sum_{i=1}^r x_i<1}}{\int\cdots\int} \prod_{i=1}^r u_i^{\alpha_i-1} \left(1-\sum_{i=1}^r x_i\right)^{\alpha_{r+1}-1} \prod_{i=1}^r \mathrm{d}x_i = \frac{\prod_{i=1}^{r+1} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{r+1} \alpha_i)}. \tag{6}$$

Furthermore, by setting $r = n-1$, $\alpha_i = a_i + j_i$, $i = 1, \ldots, n$ where $a_1, \ldots, a_n$ are positive real numbers and $j_1, \ldots, j_n$ are non-negative integers in (6), we get

$$\begin{aligned} &\underset{\substack{x_1>0,\ldots,x_{n-1}>0\\ \sum_{i=1}^{n-1} x_i<1}}{\int\cdots\int} \prod_{i=1}^{n-1} x_i^{a_i+j_i-1} \left(1-\sum_{i=1}^{n-1} x_i\right)^{a_n+j_n-1} \prod_{i=1}^{n-1} \mathrm{d}x_i \\ &\quad = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i)} \frac{(a_1)_{j_1}\cdots(a_n)_{j_n}}{(\sum_{i=1}^n a_i)_{j_1+\cdots+j_n}}. \end{aligned} \tag{7}$$

3. The Dirichlet-Lauricella Type D Distribution

The Dirichlet-Lauricella type D distribution is defined as follows.

The random variables $U_1, \ldots, U_n$ are said to have a Dirichlet-Lauricella type D distribution with parameters $a_1, \ldots, a_n; c; d; \theta_1, \ldots, \theta_n$, denoted by $(U_1, \ldots, U_n) \sim \mathrm{DLD}(a_1, \ldots, a_n; c; d; \theta_1, \ldots, \theta_n)$, if their joint p.d.f. is given by

$$K_D \frac{\prod_{i=1}^n u_i^{a_i-1} \left(1-\sum_{i=1}^n u_i\right)^{c-\sum_{i=1}^n a_i-1}}{\left(1-\sum_{i=1}^n \theta_i u_i\right)^d},\ u_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^n u_i < 1, \quad (8)$$

where $a_1 > 0, \ldots, a_n > 0$, $c - a_1 - \cdots - a_n > 0$, $-1 < \theta_1 < 1, \ldots, -1 < \theta_n < 1$ and $-\infty < d < \infty$. The normalizing constant K_D in (8) is given by

$$\begin{aligned} K_D^{-1} &= \underset{\substack{u_1>0,\ldots,u_n>0 \\ \sum_{i=1}^n u_i<1}}{\int \cdots \int} \frac{\prod_{i=1}^n u_i^{a_i-1} \left(1-\sum_{i=1}^n u_i\right)^{c-\sum_{i=1}^n a_i-1}}{\left(1-\sum_{i=1}^n \theta_i u_i\right)^d} \prod_{i=1}^n \mathrm{d}u_i \\ &= \frac{\prod_{i=1}^n \Gamma(a_i)\Gamma(c-\sum_{i=1}^n a_i)}{\Gamma(c)} F_D^{(n)}(d, a_1, \ldots, a_n; c; \theta_1, \ldots, \theta_n), \end{aligned}$$

where the last line has been obtained by using (3).

From (8), it is clear that for $d = 0$, the Dirichlet-Lauricella type D distribution reduces a Dirichlet type 1 distribution with parameters $a_1, \ldots, a_n$ and $c - \sum_{i=1}^n a_i$.

For $n = 1$, the p.d.f. in (8) simplifies to a generalized beta type 1 p.d.f. given by

$$\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)\, {}_2F_1(a_1, d; c; \theta_1)} \frac{u_1^{a_1-1}(1-u_1)^{c-a_1-1}}{(1-\theta_1 u_1)^d},\ 0 < u_1 < 1,$$

where $c > a_1 > 0$, $-1 < \theta_1 < 1$ and ${}_2F_1$ is the Gauss hypergeometric function. The generalized beta type 1 distribution defined by the above p.d.f. has been studied by Nagar and Rada-Mora [6], Nagar and Bedoya-Valencia [7], Libby and Novic [4], Pham-Gia and Duong [9].

Further, for $n = 2$, the p.d.f. in (8) slides to a generalized bivariate beta type 1 p.d.f. defined by (Nadarajah and Kotz [5]),

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)\, F_1(d, a_1, a_2; c; \theta_1, \theta_2)} \\ &\times \frac{u_1^{a_1-1} u_2^{a_2-1} (1-u_1-u_2)^{c-a_1-a_2-1}}{(1-\theta_1 u_1 - \theta_2 u_2)^d},\ u_1 > 0,\ u_2 > 0,\ u_1 + u_2 < 1, \end{aligned}$$

where $a_1 > 0$, $a_2 > 0$, $c > a_1 + a_2$, $-1 < \theta_1 < 1$, $-1 < \theta_2 < 1$ and F_1 is the first hypergeometric function of Appell.

Consider the transformation $Z_i = (1 - \sum_{j=1}^{s} U_j)^{-1} U_i$, $i = s+1, \ldots, n$. Then, $u_i = (1 - \sum_{j=1}^{s} u_j) z_i$, $i = s+1, \ldots, n$ with the Jacobian $J(u_{s+1}, \ldots, u_n \to z_{s+1}, \ldots, z_n) = (1 - \sum_{j=1}^{s} u_j)^{n-s}$. Substituting appropriately in (8), the joint density of $U_1, \ldots, U_s, Z_{s+1}, \ldots, Z_n$ is given by

$$K_D \frac{\prod_{i=1}^{s} u_i^{a_i - 1} (1 - \sum_{i=1}^{s} u_i)^{c - \sum_{i=1}^{s} a_i - 1} \prod_{i=s+1}^{n} z_i^{a_i - 1} \left(1 - \sum_{i=s+1}^{n} z_i\right)^{c - \sum_{i=1}^{n} a_i - 1}}{[1 - \sum_{i=1}^{s} \theta_i u_i - (1 - \sum_{j=1}^{s} u_j) \sum_{i=s+1}^{n} \theta_i z_i]^d}, \tag{9}$$

where $u_i > 0$, $i = 1, \ldots, s$, $\sum_{i=1}^{s} u_i < 1$, $z_i > 0$, $i = s+1, \cdots, n$, and $\sum_{i=s+1}^{n} z_i < 1$.

Now, we find the marginal p.d.f. of $U_1, \ldots, U_s$ by integrating out $z_{s+1}, \ldots, z_n$ from the joint density of $U_1, \ldots, U_s, Z_{s+1}, \ldots, Z_n$ as

$$K_D \frac{\prod_{i=1}^{s} u_i^{a_i - 1} (1 - \sum_{i=1}^{s} u_i)^{c - \sum_{i=1}^{s} a_i - 1}}{(1 - \sum_{i=1}^{s} \theta_i u_i)^d}$$
$$\times \underset{\substack{z_{s+1} > 0, \ldots, z_n > 0 \\ \sum_{i=s+1}^{n} z_i < 1}}{\int \cdots \int} \frac{\prod_{i=s+1}^{n} z_i^{a_i - 1} (1 - \sum_{i=s+1}^{n} z_i)^{c - \sum_{i=1}^{n} a_i - 1}}{[1 - (1 - \sum_{j=1}^{s} u_j) \sum_{i=s+1}^{n} \theta_i z_i / (1 - \sum_{i=1}^{s} \theta_i u_i)]^d} \prod_{i=s+1}^{n} \mathrm{d} z_i. \tag{10}$$

Now, using the integral representation of $F_D^{(n)}$, we derive the marginal p.d.f. of $U_1, \ldots, U_s$ as

$$K_{D1} \frac{\prod_{i=1}^{s} u_i^{a_i - 1} (1 - \sum_{i=1}^{s} u_i)^{c - \sum_{i=1}^{s} a_i - 1}}{(1 - \sum_{i=1}^{s} \theta_i u_i)^d}$$
$$\times F_D^{(n-s)} \left(d, a_{s+1}, \ldots, a_n; c - \sum_{i=1}^{s} a_i; \frac{\theta_{s+1} (1 - \sum_{i=1}^{s} u_i)}{1 - \sum_{i=1}^{s} \theta_i u_i}, \ldots, \frac{\theta_n (1 - \sum_{i=1}^{s} u_i)}{1 - \sum_{i=1}^{s} \theta_i u_i} \right), \tag{11}$$

where $u_i > 0$, $i = 1, \ldots, s$, $\sum_{i=1}^{s} u_i < 1$ and

$$K_{D1}^{-1} = \frac{\prod_{i=1}^{s} \Gamma(a_i) \Gamma(c - \sum_{i=1}^{s} a_i)}{\Gamma(c)} F_D^{(n)}(d, a_1, \ldots, a_n; c; \theta_1, \ldots, \theta_n). \tag{12}$$

It is interesting to observe that the marginal density of $U_1, \ldots, U_s$ does not belong to the Dirichlet-Lauricella type D family of distributions and differs by an additional factor containing the Lauricella hypergeometric function F_D.

From (11), it is straightforward to show that the marginal p.d.f. of U_s is

$$K_{D2}\frac{u_s^{a_s-1}(1-u_s)^{c-a_s-1}}{(1-\theta_s u_s)^{d_s}}F_D^{(n-1)}\left(d,a_1,\ldots,a_{s-1},a_{s+1},\ldots,a_n;c-a_s;\right.$$
$$\left.\frac{\theta_1(1-u_s)}{1-\theta_s u_s},\ldots,\frac{\theta_{s-1}(1-u_s)}{1-\theta_s u_s},\frac{\theta_{s+1}(1-u_s)}{1-\theta_s u_s},\ldots,\frac{\theta_n(1-u_s)}{1-\theta_s u_s}\right),$$

where

$$K_{D2}^{-1}=\frac{\Gamma(a_s)\Gamma(c-a_s)}{\Gamma(c)}F_D^{(n)}(d,a_1,\ldots,a_n;c;\theta_1,\ldots,\theta_n). \tag{13}$$

The marginal p.d.f. of $Z_{s+1},\ldots,Z_n$ is given by

$$K_D\frac{\prod_{i=s+1}^n z_i^{a_i-1}\left(1-\sum_{i=s+1}^n z_i\right)^{c-\sum_{i=1}^n a_i-1}}{(1-\sum_{i=s+1}^n\theta_i z_i)^d}$$
$$\times\underset{\substack{u_1>0,\ldots,u_s>0\\ \sum_{i=1}^s u_i<1}}{\int\cdots\int}\frac{\prod_{i=1}^s u_i^{a_i-1}(1-\sum_{i=1}^s u_i)^{c-\sum_{i=1}^s a_i-1}}{[1-\sum_{j=1}^s(\theta_j-\sum_{i=s+1}^n\theta_i z_i)u_j/(1-\sum_{i=s+1}^n\theta_i z_i)]^d}\prod_{i=1}^s \mathrm{d}u_i.$$

Now, evaluating the above integral by using (3), we get

$$K_{D3}\frac{\prod_{i=s+1}^n z_i^{a_i-1}\left(1-\sum_{i=s+1}^n z_i\right)^{c-\sum_{i=1}^n a_i-1}}{(1-\sum_{i=s+1}^n\theta_i z_i)^d}$$
$$\times F_D^{(s)}\left(d,a_1,\ldots,a_s;c;\frac{\theta_1-\sum_{i=s+1}^n\theta_i z_i}{1-\sum_{i=s+1}^n\theta_i z_i},\ldots,\frac{\theta_s-\sum_{i=s+1}^n\theta_i z_i}{1-\sum_{i=s+1}^n\theta_i z_i}\right),$$

where

$$K_{D3}^{-1}=\frac{\prod_{i=s+1}^n\Gamma(a_i)\Gamma(c-\sum_{i=1}^n a_i)}{\Gamma(c-\sum_{i=1}^s a_i)}F_D^{(n)}(d,a_1,\ldots,a_n;c;\theta_1,\ldots,\theta_n). \tag{14}$$

It is well known that if $(U_1,\ldots,U_n)\sim \mathrm{D1}(a_1,\ldots,a_n;c-\sum_{i=1}^n a_i)$, then

$$\left(\frac{U_1}{\sum_{i=1}^n U_i},\ldots,\frac{U_{n-1}}{\sum_{i=1}^n U_i}\right)\sim \mathrm{D1}(a_1,\ldots;a_n)$$

and the sum $\sum_{i=1}^n U_i$ follows a beta type 1 distribution with parameters $\sum_{i=1}^n a_i$ and $c-\sum_{i=1}^n a_i$. In the next theorem, we derive similar result for the Dirichlet-Lauricells type D distribution.

Theorem 3.1. *Let* $(U_1, \dots, U_n) \sim \text{DLD}(a_1, \dots, a_n; c; d; \theta_1, \dots, \theta_n)$ *and define* $U = \sum_{i=1}^{n} U_i$ *and* $X_i = U_i/U$, $i = 1, \dots, n-1$. *Then, the joint p.d.f. of* $X_1, \dots, X_{n-1}$ *is given as*

$$K_D \frac{\Gamma(\sum_{i=1}^{n} a_i)\Gamma(c - \sum_{i=1}^{n} a_i)}{\Gamma(c)} \prod_{i=1}^{n-1} x_i^{a_i - 1} \left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n - 1}$$
$$\times {}_2F_1\left(\sum_{i=1}^{n} a_i, d; c; \sum_{i=1}^{n-1} \theta_i x_i + \theta_n \left(1 - \sum_{i=1}^{n-1} x_i\right)\right),$$

where $x_i > 0$, $i = 1, \dots, n-1$, $\sum_{i=1}^{n-1} x_i < 1$. *Further, the p.d.f. of* U *is derived as*

$$K_D \frac{\prod_{i=1}^{n} \Gamma(a_i)}{\Gamma(\sum_{i=1}^{n} a_i)} u^{\sum_{i=1}^{n} a_i - 1} (1-u)^{c - \sum_{i=1}^{n} a_i - 1}$$
$$\times F_D^{(n)}\left(d, a_1, \dots, a_n; \sum_{i=1}^{n} a_i; \theta_1 u, \dots, \theta_n u\right), \ 0 < u < 1.$$

Proof. Substituting $u_i = u x_i$, $i = 1, \dots, n-1$ and $u_n = u(1 - \sum_{i=1}^{n-1} x_i)$ with the Jacobian $J(u_1, \dots, u_n \to x_1, \dots, x_{n-1}, u) = u^{n-1}$ in the p.d.f. of $(U_1, \dots, U_n)$, we get the joint p.d.f. of $(X_1, \dots, X_{n-1})$ and U as

$$K_D \prod_{i=1}^{n-1} x_i^{a_i - 1} \left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n - 1} \frac{u^{\sum_{i=1}^{n} a_i - 1}(1-u)^{c - \sum_{i=1}^{n} a_i - 1}}{[1 - \sum_{i=1}^{n-1} \theta_i x_i u - \theta_n u(1 - \sum_{i=1}^{n-1} x_i)]^d},$$

where $x_i > 0$, $i = 1, \dots, n-1$, $\sum_{i=1}^{n-1} x_i < 1$ and $0 < u < 1$. Now, integrating u in the above expression by using the integral representation of ${}_2F_1$, we get the desired result. Further, writing

$$\left[1 - \sum_{i=1}^{n-1} \theta_i x_i u - \theta_n u \left(1 - \sum_{i=1}^{n-1} x_i\right)\right]^{-d}$$
$$= \sum_{j_1, \dots, j_n = 0}^{\infty} (d)_{j_1 + \dots + j_n} \frac{(\theta_1 x_1 u)^{j_1} \cdots (\theta_{n-1} x_{n-1} u)^{j_{n-1}} [\theta_n (1 - \sum_{i=1}^{n-1} x_i) u]^{j_n}}{j_1! \cdots j_n!}$$

the joint p.d.f. of $(X_1, \dots, X_{n-1})$ and U is rewritten as

$$K_D u^{\sum_{i=1}^{n} a_i - 1} (1-u)^{c - \sum_{i=1}^{n} a_i - 1} \sum_{j_1, \dots, j_n = 0}^{\infty} (d)_{j_1 + \dots + j_n}$$

$$\times \frac{(\theta_1 u)^{j_1} \cdots (\theta_{n-1} u)^{j_{n-1}} (\theta_n u)^{j_n}}{j_1! \cdots j_n!} \prod_{i=1}^{n-1} x_i^{a_i+j_i-1} \left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n+j_n-1}.$$

Now, integrating the above expression with respect to $x_1, \ldots, x_{n-1}$ by using (7) and summing the resulting series by applying (2), we get the desired result. □

By definition, the product moments are obtained as

$$\begin{aligned} E\left[\prod_{i=1}^{n} u_i^{r_i}\right] &= \int \cdots \int_{\substack{u_1>0,\ldots,u_n>0 \\ \sum_{i=1}^n u_i<1}} \frac{\prod_{i=1}^n u_i^{a_i+r_i-1} (1-\sum_{i=1}^n u_i)^{c-\sum_{i=1}^n a_i -1}}{(1-\sum_{i=1}^n \theta_i u_i)^d} \prod_{i=1}^n \mathrm{d}u_i \\ &= \frac{\Gamma(c)\prod_{i=1}^n \Gamma(a_i+r_i)}{\Gamma(c+r)\prod_{i=1}^n \Gamma(a_i)} \\ &\quad \times \frac{F_D^{(n)}(d, a_1+r_1, \ldots, a_n+r_n; c+r; \theta_1, \ldots, \theta_n)}{F_D^{(n)}(d, a_1, \ldots, a_n; c; \theta_1, \ldots, \theta_n)}, \end{aligned}$$

where $r = \sum_{i=1}^n r_i$, $\mathrm{Re}(a_i + r_i) > 0$, $i = 1, \ldots, n$ and $\mathrm{Re}(c + r) > 0$. Further

$$\begin{aligned} E\left[\left(1-\sum_{i=1}^{n} u_i\right)^h\right] &= \int \cdots \int_{\substack{u_1>0,\ldots,u_n>0 \\ \sum_{i=1}^n u_i<1}} \frac{\prod_{i=1}^n u_i^{a_i-1} (1-\sum_{i=1}^n u_i)^{c+h-\sum_{i=1}^n a_i -1}}{(1-\sum_{i=1}^n \theta_i u_i)^d} \prod_{i=1}^n \mathrm{d}u_i \\ &= \frac{\Gamma(c)\Gamma(c+h-\sum_{i=1}^n a_i)}{\Gamma(c+h)\Gamma(c-\sum_{i=1}^n a_i)} \\ &\quad \times \frac{F_D^{(n)}(d, a_1, \ldots, a_n; c+h; \theta_1, \ldots, \theta_n)}{F_D^{(n)}(d, a_1, \ldots, a_n; c; \theta_1, \ldots, \theta_n)}. \end{aligned}$$

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