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PI-1-1 NORMAL FORM IN A REGULAR CARDINAL

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Abstract: A normal form is given for Π^1_1 formulas over V_κ for a regular cardinal κ. Using it, a new characterization of weakly compact cardinals is given.

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1. Formulas

Normal form theorems for definable sets of reals are central to descriptive set theory. Here such will be considered for definable sets over V_κ where κ is a regular cardinal. In the case $\kappa = \omega$, the Ackerman coding [1] provides a definable bijection $V_\omega \mapsto \omega$, so V_ω could be used as the basic domain in this case.

Let $L_\in$ denote the first order language of set theory, with binary relations $=, \in$. Let $L^s_\in$ denote $L_\in$ with set variables added, considered as unary predicate symbols (so $X(x)$ is an atomic formula). Let $L^f_\in$ denote $L_\in$ with function variables added, considered as unary function symbols (so $F(x)$ is a term).

For each of these languages, a bounded quantifier is one of the form $\forall x \in t$ or $\exists x \in t$, where t is a term (a variable in the case of $L_\in$ or $L^s_\in$), such that x does not occur in t. A bounded, or Δ^0_0, formula is a formula where all quantifiers

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are bounded. For $n \geq 1$ a Σ^0_n (Π^0_n) formula is a Δ^0_0 formula, preceded by n alternating blocks of first-order quantifiers, the first of which is $\exists$ ($\forall$).

A Δ^1_0 formula is a formula which is Δ^0_0, Σ^0_n for some n, or Π^0_n for some n. For $n \geq 1$ a Σ^1_n (Π^1_n) formula is a Δ^1_0 formula, preceded by n alternating blocks of second-order quantifiers, the first of which is $\exists$ ($\forall$). For $j = 1, 2$ the notation Σ^{js}_n is used for "Σ^j_1 over $L^s_\in$", etc.

Let Pow(x) denote the power set of x. Let $\mathcal{C}$ denote Pow(V_κ). Formulas of $L^s_\in$ may be considered as defining predicates in a Cartesian product of copies of V_κ and $\mathcal{C}$ (a "product space"). Let $\mathcal{N}$ denote $(V_\kappa)^{V_\kappa}$. Formulas of $L^f_\in$ may be considered as defining predicates in a Cartesian product of copies of V_κ and $\mathcal{N}$.

Say that a predicate is Σ^j_n for $j = 1, 2$ (Π^j_n) if it is defined by a Σ^j_n (Π^j_n) formula. By predicate logic:

- Any of these classes is closed under $\wedge$ and $\vee$.
- If P is Σ^j_n (Π^j_n) then $\neg P$ is Π^j_n (Σ^j_n)
- If $m > n$ then a Σ^j_n or Π^j_n predicate is both Σ^j_m and Π^j_m.

Various other closure properties may be shown; see [3] for example.

The notion of a Δ^j_n predicate is defined in the usual manner, and by a standard abuse of language, a Δ^j_n formula is a Σ^j_n or Π^j_n formula which defines a Δ^j_n predicate.

2. $L^s_\in$ and $L^f_\in$

Lemma 1. *For any atomic formula ϕ of $L^f_\in$ there is a Δ^0_1 formula ϕ' of $L^s_\in$ such that $\phi(F_1, \ldots, F_k)$ iff $\phi'(\gamma_{F_1}, \ldots, \gamma_{F_k})$ where γ_F is the graph of F.*

Proof. In a standard manner, $t = u$ or $t \in u$ may be rewritten in a Δ^0_1 form involving only atomic formulas of the form $y = F(x)$, $w = x$, or $w \in x$. Now, $y = F(x)$ iff $\exists p(p = \langle x, y\rangle \wedge \gamma_F(p))$ iff $\forall p(p = \langle x, y\rangle \Rightarrow \gamma_F(p))$. Finally, $p = \langle x, y\rangle$ is Δ^0_0. □

Lemma 2. *The predicate Func(X) stating that X is the graph of a function is Π^{0s}_2.*

Proof. The formula $\forall x \exists p \exists y(p = \langle x, y\rangle \wedge X(p))$ states that X is total. The formula $\forall x \forall p_1 \forall p_2 \forall y_1 \forall y_2 (p_1 = \langle x, y_1\rangle \wedge p_2 = \langle x, y_2\rangle \wedge X(p_1) \wedge X(p_2) \Rightarrow p_1 = p_2)$ states that X is single-valued. □

Lemma 3. *For any atomic formula ϕ of $L^s_\in$ there is a Δ^0_0 formula ϕ' of $L^f_\in$ such that $\phi(X_1, \ldots, X_k)$ iff $\phi'(\chi_{X_1}, \ldots, \chi_{X_k})$ where χ_X is the characteristic function of X.*

Proof. $X(x)$ iff $\chi_X(x) \neq 0$. □

Further facts can be stated; see for example theorems 15.XXV and 15.XXVI of [6] in the case of ω.

3. Normal Form

As in the case of ω, a normal form is more useful for formulas of $L^f_\in$ (see [5] for example). In this case Skolem functions can be expressed in a simple manner. The notation $\vec{x}$ is used to denote a sequence $x_1, \ldots, x_k$ of variables; $\exists\vec{x}$ denotes $\exists x_1 \cdots \exists x_k$, etc. A formula ϕ with free variables or parameters among $\vec{F}, \vec{x}$ may be denoted $\phi(\vec{F}, \vec{x})$.

Theorem 4. *If ϕ is a Δ^{1f}_0 formula there is a Δ^{0f}_0 formula ψ such that ϕ is equivalent to $\exists\vec{F}\forall\vec{x}\psi$ (in models of a suitable fragment of set theory for $L^f_\in$).*

Proof. This is a refinement of the Skolem normal form theorem of Section 4.2 of [4]. Indeed, by this theorem ϕ can be written as $\exists F_1 \cdots \exists F_k \forall x_1 \cdots \forall x_l \psi_1$ where in ψ_1 F_i may have multiple arguments (where in fact ψ_1 can be taken as open).

This may be rewritten as $\exists\vec{F}\forall\vec{x}\forall\vec{y}(H_1 \wedge \cdots \wedge H_k \Rightarrow \psi_2)$, where H_i is $y_i = F_i(\vec{x}_i)$, $\vec{x}_i$ is a subsequence of $\vec{x}$, and ψ_2 is ψ_1 with $F_i(\vec{x}_i)$ replaced by y_i.

Let H'_i be $x_{j_1} = 0 \wedge \cdots \wedge x_{j_t} \Rightarrow y_i = G_i(z)$, where the x_j are those not occurring in $\vec{x}$. The formula may be transformed to $\exists\vec{G}\forall\vec{x}\vec{y}z(z = \langle x_1, \ldots, x_l\rangle \wedge H'_1 \wedge \cdots \wedge H'_k \Rightarrow \psi_2)$. □

The models in which the theorem holds include V_κ for κ a cardinal.

Corollary 5. *If ϕ is a Σ^{1f}_1 formula there is a Δ^{0f}_0 formula ψ such that ϕ is equivalent to $\exists\vec{F}\forall\vec{x}\psi$.*

Proof. Transform the first order part to Skolem normal form, prepend the original existential function quantifiers, and proceed as in the proof of the theorem. □

Corollary 6. *If ϕ is a Π^{1f}_1 formula there is a Δ^{0f}_0 formula ψ such that ϕ is equivalent to $\forall\vec{F}\exists\vec{x}\psi$.*

Proof. Apply the preceding corollary to $\neg\phi$. □

$\exists\vec{F}\forall\vec{x}\psi$ may be transformed to $\exists F\forall x\psi$. Proceeding as in the proof of theorem 4, $\exists\vec{G}\forall\vec{x}\vec{y}z(z = \langle x_1, \dots, x_l\rangle \wedge H'_1 \wedge \cdots \wedge H'_k \Rightarrow \psi_2)$ may be further transformed to $\exists G\forall\vec{x}\vec{y}\vec{w}zv(v = G(z) \wedge v = \langle w_1, \dots, w_k\rangle \wedge z = \langle x_1, \dots, x_l\rangle \wedge H''_1 \wedge \cdots \wedge H''_k \Rightarrow \psi_2)$, where H''_i is $x_{j_1} = 0 \wedge \cdots \wedge x_{j_t} \Rightarrow y_i = w_i$. Finally, a formula $\forall\vec{x}\psi$ may be transformed to $\forall u\forall\vec{x}(u = \langle x_1, \dots, x_l\rangle \Rightarrow \psi)$. The quantifier $\forall x_i$ may then be replaced by bounded quantifiers.

Suppose κ is a regular cardinal, $\phi(\vec{F})$ is a Δ^{0f}_0 formula over V_κ (possibly with first order parameters), and $u \in V_\kappa$. Let $\phi^{\restriction u}$ denote ϕ with each occurrence of F_i replaced by $F_i \restriction u$. This may be written in Δ^{0f}_1 form in a standard manner, as $\exists\vec{f}(f_1 = F_1 \restriction u \wedge \cdots \wedge f_k = F_k \restriction u \wedge \phi(\vec{f}))$ and $\forall\vec{f}(f_1 = F_1 \restriction u \wedge \cdots \wedge f_k = F_k \restriction u \Rightarrow \phi(\vec{f}))$.

For a Δ^{0f}_0 formula ψ with free variables $\vec{x}, \vec{F}$ a predicate $Q_\psi(u, \vec{x}, \vec{F})$ will be defined by recursion on ψ, which holds iff the arguments of function applications are all in u. For a term t let Q_t is $\vee_s s \in u$ where s ranges over subterms other than t. $Q_{t=u} = Q_{t\in u} = Q_t \wedge Q_u$, $Q_{\neq\psi} = Q_\psi$, $Q_{\psi_1\wedge\psi_2} = Q_{\psi_1\vee\psi_2} = Q_{\psi_1} \wedge Q_{\psi_2}$, and $Q_{\exists x\in t\psi} = Q_{\forall x\in t\psi} = Q_t \wedge \forall x \in tQ_\psi$.

Lemma 7. *Suppose κ is a cardinal; then in V_κ, $Q_\psi \Rightarrow (\psi \Leftrightarrow \psi^{\restriction u})$. If κ is regular, $\exists uQ_\psi$.*

Proof. The first claim follows by induction on ψ. The second claim does also, using regularity for the bounded quantifiers. □

4. Downsets

The notion of a tree is central to descriptive set theory. The generalization to a cardinal κ is better behaved if κ is assumed to be regular. Let $\mathcal{A} = \{f \in V_\kappa : f$ is a function$\}$; by the restriction on κ if a function $f : x \mapsto V_\kappa$ where $x \in V_\kappa$ then $f \in V_\kappa$.

$\mathcal{A}$ is ordered by inclusion. A subset $D \subseteq \mathcal{A}$ is said to be a downset if, $f \in D$ and $g \subseteq f$ imply $g \in D$. If D is a downset a branch of D is a function $F \in \mathcal{N}$ such that $F \restriction x \in D$ for all $x \in V_\kappa$. Let $[D]$ denote the set of branches of D. If $[D] \neq \emptyset$ say that D is branched; otherwise $[D]$ is unbranched.

$\mathcal{A}$ may be generalized in a well-known manner. Let $\mathcal{A}^{(k)} = \{\vec{f} \in \mathcal{A}^k : \mathrm{Dom}(f_1) = \cdots = \mathrm{Dom}(f_k)\}$ where $\mathrm{Dom}(f)$ denotes the domain of the function f; $\mathrm{Dom}(\vec{f})$ may be written for the common domain. $\mathcal{A}^{(k)}$ may be ordered by "componentwise inclusion" $\vec{f} \subseteq_c \vec{g}$, which holds iff $f_i \subseteq g_i$ for all i. The definition of a downset is essentially unchanged, and a branch is a vector $\vec{F}$ which is a branch componentwise.

If D is a downset in $\mathcal{A}^{k+l}$ and $\vec{G} \in \mathcal{N}^l$ let $D_{@\vec{G}} = \{\vec{f} : \langle \vec{f}, G_1 \restriction \mathrm{Dom}(\vec{f}), \ldots, G_l \restriction \mathrm{Dom}(\vec{f}) \rangle \in D\}$. Note that $D_{@\vec{G}}$ is a downset.

A Δ_0^{0f} formula $\psi(\vec{F})$ may be translated to a Δ_0^{1f} formula $\psi'(\vec{f})$ in a well-known manner; by abuse of notation $\psi(\vec{f})$ will be written for this formula. The notation "$\vec{x} \in_c u$" will be used for $x_1 \in u \wedge \cdots \wedge x_t \in u$.

Theorem 8. *Suppose κ is a regular cardinal. Suppose ϕ is a formula $\forall \vec{F} \exists \vec{x} \psi(\vec{x}, \vec{F}, \vec{G})$ where ψ is Δ_0^{0f} (and for brevity has no first order parameters). Let*

$$D = \{\langle \vec{f}, \vec{g} \rangle \in \mathcal{A}^{k+l} : \forall \vec{x} \in_c Dom(\vec{f})(Q_\psi(Dom(\vec{f}), \vec{x}, \vec{f}, \vec{g}) \Rightarrow \neg\psi(\vec{x}, \vec{f}, \vec{g}))\}.$$

Then D is a downset, and in V_κ, ϕ holds iff $D_{@\vec{G}}$ is unbranched.

Proof. It follows readily by induction on ψ that if $\langle \vec{f}', \vec{g}' \rangle \subseteq_c \langle \vec{f}, \vec{g} \rangle$ then $Q_\psi(\mathrm{Dom}(\vec{f}'), \vec{x}, \vec{f}', \vec{g}') \Rightarrow Q_\psi(\mathrm{Dom}(\vec{f}), \vec{x}, \vec{f}, \vec{g})$. It then follows that D is a downset. Using lemma 7 it follows readily that if ϕ is true then $D_{@\vec{G}}$ is unbranched; and conversely. ☐

5. Weak Compactness

It is a well-known fact that a cardinal κ is weakly compact iff it is Π_1^1-indescribable, that is, for any Π_1^{1s} sentence ϕ, if $\models_{V_\kappa} \phi$ then $\models_{V_\lambda} \phi$ for some regular cardinal $\lambda < \kappa$.

Theorem 9. *For a regular cardinal κ the following are equivalent.*

a. *For any Π_1^{1f} sentence $\forall F \psi(F, G)$, if $\models_{V_\kappa} \forall F \psi(F, G)$ then for some regular cardinal $\lambda < \kappa$, $G \cap V_\lambda$ is total and $\models_{V_\lambda} \forall F \psi(F, G \cap V_\lambda)$.*
b. *For any Π_1^{1f} sentence $\forall F \psi(F, G)$ where $G \cap V_\lambda$ is total for all regular cardinals $\lambda < \kappa$, if $\models_{V_\kappa} \forall F \psi(F, G)$ then for some regular cardinal $\lambda < \kappa$, $\models_{V_\lambda} \forall F \psi(F, G \cap V_\lambda)$.*
c. *κ is weakly compact.*

Proof. a⇒b is immediate. Suppose b holds. Suppose ϕ is a Π_1^{1s} formula $\forall X \psi(X, Y)$. Let ϕ' be $\forall F \psi'(F, \chi_Y)$ where ψ' is the translation of ψ obtained using lemma 3. That b⇒c follows, noting that $\chi_Y \cap V_\lambda$ is total.

Suppose κ is weakly compact. Given $\phi = \forall F \psi(F, G)$, let ϕ' be $\mathrm{Tot}(\gamma_G) \wedge \forall X(\mathrm{Func}(X) \Rightarrow \psi')$ where ψ' is the translation of ψ obtained using lemma 1, Func is as in lemma 2, and Tot(X) is the first order predicate "X is total". Then $\models_{V_\kappa} \phi'$, so $\models_{V_\lambda} \phi'$ for some λ, so $G \cap V_\lambda$ is total and $\models_{V_\lambda} \phi$. Thus, c⇒a. □

Say that a regular cardinal κ has the downset property iff, whenever $D \subseteq \mathcal{N}$ is an unbranched downset, there is a regular cardinal $\lambda < \kappa$ such that $D \cap V_\lambda$ is unbranched.

Theorem 10. *If a regular cardinal κ has the downset property then κ is weakly compact.*

Proof. Suppose ϕ is a Π^{1f}_1 formula, which may be assumed to be in normal form $\forall\vec{F}\exists\vec{x}\psi$ where ψ is Δ^{1f}_1, has no first order parameters, and for a second order parameter G and a regular cardinal κ $G \cap V_\lambda$ is total. Suppose ϕ is true in V_κ. Then the downset $D_{@\vec{G}}$ of theorem 8 is unbranched. By hypothesis $D_{@\vec{G}} \cap V_\lambda$ is unbranched for some regular cardinal $\lambda < \kappa$. By theorem 8 ϕ is true in V_λ. □

Theorem 11. *If κ is a weakly compact cardinal then κ has the downset property.*

Proof. There is a Π^{1f}_1 sentence in the parameter D which is true in V_λ for a regular cardinal λ iff D is an unbranched downset. □

6. Concluding Remarks

It is a question of considerable interest, whether the existence of weakly compact cardinals can be justified by postulating sufficiently long stationary set chains (see[2]). Although the results given here do not shed much light on the problem, they do indicate that methods from descriptive set theory can be adopted to some extent. This suggests that further research in this area might be of interest.

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