THE MONOPHONIC GRAPHOIDAL COVERING NUMBER OF A GRAPH

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Abstract: A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A monophonic graphoidal cover of a graph $G$ is a collection $\psi_m$ of monophonic paths in $G$ such that every vertex of $G$ is an internal vertex of at most one monophonic path in $\psi_m$ and every edge of $G$ is in exactly one monophonic path in $\psi_m$. The minimum cardinality of a monophonic graphoidal cover of $G$ is called the monophonic graphoidal covering number of $G$ and is denoted by $\eta_m$. We determine bounds for it and characterize graphs which realize these bounds. Also, for any positive integer $n$ with $q - p + 2 \leq n \leq q - 1$, there exists a tree $T$ such that the monophonic graphoidal covering number is $n$.

AMS Subject Classification: 05C70

Key Words: graphoidal cover, acyclic graphoidal cover, geodesic graphoidal cover, monophonic path, monophonic graphoidal cover, monophonic graphoidal covering number

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary[6]. The concept of graphoidal cover was introduced by Acharya and Sampathkumar[2] and further studied in [1, 3, 7, 8].

A graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions.

(i) Every path in $\psi$ has at least two vertices.

(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.

(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$.

The collection $\psi$ is called an acyclic graphoidal cover of $G$ if no member of $\psi$ is cycle; it is called a geodesic graphoidal cover if every member of $\psi$ is a shortest path in $G$. The minimum cardinality of an acyclic (geodesic) graphoidal cover of $G$ is called the acyclic (geodesic) graphoidal covering number of $G$ and is denoted by $\eta_a(\eta_g)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4, 5].

A chord of a path $P$ is an edge joining any two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. For any two vertices $u$ and $v$ in a connected graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is defined as the length of a longest $u - v$ monophonic path in $G$. The monophonic eccentricity $e_m(v)$ of a vertex $v$ in $G$ is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The monophonic radius is $rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter is $diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [9, 10].

The following theorems will be used in the sequel.

**Theorem 1.1.** [6] Every non-trivial connected graph has at least two vertices which are not cut vertices.

**Theorem 1.2.** [6] Let $G$ be a connected graph with at least three vertices. The following statements are equivalent:

(i) $G$ is a block

(ii) Every two vertices of $G$ lie on a common cycle.

**Theorem 1.3.** [5] Let $K_{m,n}(1 \leq m \leq n)$ be a bipartite graph. Then
\[
\eta_g(K_{m,n}) = \begin{cases} 
1 & \text{if } m = 1, n = 1 \\
 n - 1 & \text{if } m = 1, n \geq 2 \\
 n & \text{if } m = 2, n \geq 2 \\
 m + n - 1 & \text{if } m = 3, n = 3, 4 \\
 m + n & \text{if } m = 3, n = 5 \\
 2n - 3 & \text{if } m = 3, n \geq 6 \\
 mn - m - n & \text{if } m, n \geq 4.
\end{cases}
\]

Throughout this paper \(G\) denotes a connected graph with at least two vertices.

2. Monophonic Graphoidal Cover

**Definition 2.1.** A *monophonic graphoidal cover* of a graph \(G\) is a collection \(\psi_m\) of monophonic paths in \(G\) such that every vertex of \(G\) is an internal vertex of at most one monophonic path in \(\psi_m\) and every edge of \(G\) is in exactly one monophonic path in \(\psi_m\). The minimum cardinality of a monophonic graphoidal cover of \(G\) is called the *monophonic graphoidal covering number* of \(G\) and is denoted by \(\eta_m(G)\).

**Example 2.2.** For the graph \(G\) given in Figure 2.1, \(\psi_m = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7), (v_3, v_{10}, v_1, v_8, v_7, v_9, v_5)\}\) is a minimum monophonic graphoidal cover of \(G\) and so \(\eta_m(G) = 2\).

![Figure 2.1: G](image)

**Theorem 2.3.** For any connected graph \(G\), \(\eta(G) \leq \eta_a(G) \leq \eta_m(G) \leq \eta_g(G)\).

**Proof.** Since any acyclic graphoidal cover is a graphoidal cover and any monophonic graphoidal cover is an acyclic graphoidal cover, we have \(\eta(G) \leq \eta_a(G) \leq \eta_m(G)\). Also, since every geodesic is a monophonic path, we have every
geodesic graphoidal cover is a monophonic graphoidal cover and so \( \eta_m(G) \leq \eta_g(G) \). Hence \( \eta(G) \leq \eta_a(G) \leq \eta_m(G) \leq \eta_g(G) \).

Remark 2.4. For the graph \( K_2 \), \( \eta(K_2) = \eta_a(K_2) = 1 \), for the cycle \( C_5 \), \( \eta_a(C_5) = \eta_m(C_5) = 2 \), for the cycle \( C_3 \), \( \eta_m(C_3) = \eta_g(C_3) = 3 \). Further, for a tree \( T \), \( \eta(T) = \eta_a(T) = \eta_m(T) = \eta_g(T) = n - 1 \), where \( n \) is the number of end vertices of \( T \). All the inequalities in Theorem 2.3 can be strict. For the graph \( G \) given in Figure 2.2, \( \eta(G) = 2 \), \( \eta_a(G) = 3 \), \( \eta_m(G) = 4 \) and \( \eta_g(G) = 5 \). Thus we have \( \eta(G) < \eta_a(G) < \eta_m(G) < \eta_g(G) \).

![Figure 2.2: G](image)

Since \( q - p \leq \eta_a(G) \leq q \) and \( \eta_a(G) \leq \eta_m(G) \leq q \), we have \( q - p \leq \eta_m(G) \leq q \).

Now, we proceed to characterize graphs \( G \) for which the bounds of \( \eta_m(G) \) are attained.

For any monophonic graphoidal cover \( \psi_m \) of a graph \( G \), let \( t_{\psi_m} \) denote the number of vertices of \( G \) which are not internal vertices of any path in \( \psi_m \). Let \( t_m = \min t_{\psi_m} \), where the minimum is taken over all graphoidal covers of \( G \).

**Theorem 2.5.** For any graph \( G \), \( \eta_m(G) = q - p + t_m \).

**Proof.** Let \( \psi_m \) be any monophonic graphoidal cover of \( G \). Then \( q = \sum_{P \in \psi_m} |E(P)| = |\psi_m| + \sum_{P \in \psi_m} t_m(P) = |\psi_m| + p - t_{\psi_m} \). Therefore \( |\psi_m| = q - p + t_{\psi_m} \). Since \( \eta_m(G) \) is the minimum cardinality of a monophonic graphoidal cover of \( G \), we have \( \eta_m(G) = q - p + t_m \). \( \square \)

**Corollary 2.6.** Let \( T \) be a tree with \( n \) pendant vertices. Then \( \eta_m(T) = n - 1 \).

**Corollary 2.7.** Let \( G \) be a graph having \( n \) simplicial vertices. Then \( \eta_m(G) \geq q - p + n \). Furthermore, equality holds if and only if there exists a monophonic graphoidal cover \( \psi_m \) of \( G \) such that every non-simplicial vertex of \( G \) is an internal vertex of a unique monophonic path in \( \psi_m \).

The following proposition is the characterization result of the lower bound of \( \eta_m(G) \) and it follows from Corollary 2.7.
Proposition 2.8. For any connected graph \( G \) of order at least 3, \( \eta_m(G) = q - p \) if and only if \( G \) has no simplicial vertices and there exists a monophonic graphoidal cover \( \psi_m \) such that every vertex of \( G \) is an internal vertex of a unique monophonic path in \( \psi_m \).

Theorem 2.9. For any connected graph \( G \), \( \eta_m(G) = q \) if and only if \( G \) is complete.

Proof. Let \( G \) be a complete graph. Since any two vertices of \( G \) are adjacent, the length of any monophonic path is one. Hence \( E(G) \) is the unique monophonic graphoidal cover of \( G \) and so \( \eta_m(G) = q \).

Conversely, suppose that \( \eta_m(G) = q \). Claim that \( G \) is complete. If \( G \) is not complete, then there exists a monophonic path, say \( P \), in \( G \) such that \( |E(P)| > 1 \). Then \( \psi_m = \{E(G) - E(P)\} \cup \{P\} \) is a monophonic graphoidal cover of \( G \) and so \( \eta_m(G) \leq q - 1 \), which is a contradiction. \( \square \)

Theorem 2.10. For any connected graph \( G \) of order \( p \geq 3 \), \( \eta_m(G) = q - 1 \) if and only if \( G = K_1 + \bigcup m_jK_j \), where \( \sum m_j \geq 2 \).

Proof. Let \( \eta_m(G) = q - 1 \). Since \( p \geq 3 \), by Theorem 1.1 there exists a vertex \( x \), which is not a cut vertex of \( G \). If \( G \) has two or more cut vertices, then let \( P \) be a monophonic path containing at least two cut vertices. Then \( |E(P)| \geq 3 \). Clearly, \( \psi_m = \{E(G) - E(P)\} \cup \{P\} \) is a monophonic graphoidal cover of \( G \) and so \( \eta_m(G) \leq |\psi_m| = q - 2 \), which is a contradiction. Thus the number of cut vertices \( k \) of \( G \) is at most one.

Case (i): If \( k = 0 \), then the graph \( G \) is a block. If \( p = 3 \), then \( G = K_3 \) and so by Theorem 2.9, \( \eta_m(G) = q \), which is a contradiction to the assumption. If \( p \geq 4 \), we claim that \( G \) is complete. Suppose that \( G \) is not complete. Then there exists two vertices \( x \) and \( y \) in \( G \) such that \( d(x,y) \geq 2 \). By Theorem 1.2, \( x \) and \( y \) lie on a common cycle and hence \( x \) and \( y \) lie on a smallest cycle \( C = x, x_1, x_2, ..., y, ..., x_n, x \) of length at least 4. Clearly, all the edges of \( C \) lie on either an \( x-y \) monophonic path, say \( P_1 \), or an \( y-x \) monophonic path, say \( P_2 \). Then \( \psi_m = \{E(G) - E(C)\} \cup \{P_1, P_2\} \) is a monophonic graphoidal cover of \( G \) and so \( \eta_m(G) \leq q - 2 \), which is a contradiction. Hence \( G \) is complete and so by Theorem 2.9, \( \eta_m(G) = q \), which is again a contradiction. Thus \( k \neq 0 \).

Case (ii): If \( k = 1 \), let \( x \) be the cut vertex of \( G \). If \( p = 3 \), then \( G = P_3 = K_1 + \bigcup m_jK_1 \) where \( \sum m_j = 2 \). If \( p \geq 4 \), we claim that \( G = K_1 + \bigcup m_jK_j \), \( \sum m_j \geq 2 \). It is enough to prove that every block of \( G \) is complete. Suppose that there exists a block \( B \), which is not complete. Let \( u \) and \( v \) be two vertices in \( B \) such that \( d(u,v) \geq 2 \). Then as in Case (i), \( \eta_m(G) \leq q - 2 \), which is a
contradiction. Thus every block of $G$ is complete so that $G = K_1 + \cup m_j K_j$, where $K_1$ is the vertex $x$ and $\sum m_j \geq 2$.  

**Theorem 2.11.** For any cycle $C_p$ ($p \geq 4$), $\eta_m(C_p) = 2$.

**Proof.** Let $C_p : v_1, v_2, v_3, ..., v_p, v_1$ be a cycle of order $p$. Then $\psi_m = \{(v_1, v_2, v_3), (v_3, v_4, ..., v_p, v_1)\}$ is a minimum monophonic graphoidal cover of $C_p$ and hence $\eta_m(C_p) = 2$.  

Since every monophonic path in $K_{m,n}$ is a geodesic, we have the following result by Theorem 1.3.

**Theorem 2.12.** Let $K_{m,n}(1 \leq m \leq n)$ be a bipartite graph. Then

$$
\eta_m(K_{m,n}) = \begin{cases} 
1 & \text{if } m = 1, n = 1 \\
n - 1 & \text{if } m = 1, n \geq 2 \\
n & \text{if } m = 2, n \geq 2 \\
m + n - 1 & \text{if } m = 3, n = 3, 4 \\
m + n & \text{if } m = 3, n = 5 \\
2n - 3 & \text{if } m = 3, n \geq 6 \\
mn - m - n & \text{if } m, n \geq 4.
\end{cases}
$$

**Theorem 2.13.** Let $G$ be a unicyclic graph with $n$ pendant vertices. Let $C$ be the unique cycle in $G$ having length greater than 3 and let $k$ be the number of vertices of degree greater than 2 on $C$. Then

$$
\eta_m(G) = \begin{cases} 
2 & \text{if } k = 0 \\
n & \text{if there exists two non-adjacent vertices of degree } > 2 \text{ on } C \text{ (or) all vertices in } C \text{ are of degree } > 2 \\
n + 1 & \text{otherwise.}
\end{cases}
$$

**Proof.** Let $C : v_0, v_1, v_2, ..., v_l, v_0$ be the unique cycle in $G$ having length greater than 3.

Case (i): $k = 0$. Then $G = C$ and by Theorem 2.11, $\eta_m(G) = 2$.

Case (ii): $k = 1$. Let $v_0$ (say) be the unique vertex of degree greater than 2 on $C$. Let $G' = G - \{v_1\}$. Then $G'$ is a tree with $n + 1$ pendant vertices and hence by Corollary 2.6, $\eta_m(G') = n$. Let $\psi'_m$ be a minimum monophonic graphoidal cover of $G'$. Clearly any path in $\psi'_m$ is a monophonic path in $G$, we have $\psi_m = \psi'_m \cup \{(v_0, v_1, v_2)\}$ is a monophonic graphoidal cover of $G$. Hence $\eta_m(G) \leq n + 1$. 

Also, at least one vertex on \( C \) and all the \( n \) pendant vertices are exterior vertices of any minimum monophonic graphoidal cover of \( G \), we have \( t_m \geq n + 1 \). Then by Theorem 2.5, \( \eta_m(G) = q - p + t_m \geq n + 1 \). Hence \( \eta_m(G) = n + 1 \).

Case (iii): \( k = 2 \) and the vertices of degree greater than \( 2 \) on \( C \) are adjacent in \( G \).

Let \( v_0, v_1 \) be vertices of degree greater than \( 2 \) on \( C \). Let \( P = (v_1, v_2, v_3) \) be a \( v_1 - v_3 \) monophonic path in \( G \). Let \( G' \) be the subgraph obtained by deleting \( v_2 \) from \( G \). Clearly \( G' \) is a tree with \( n + 1 \) pendant vertices and hence by Corollary 2.6, \( \eta_m(G') = n \). If \( \psi'_m \) is a minimum monophonic graphoidal cover of \( G' \), then \( \psi'_m \cup \{P\} \) is a monophonic graphoidal cover of \( G \) and hence \( \eta_m(G) \leq n + 1 \). Also, at least one vertex on \( C \) and all the \( n \) pendant vertices are exterior vertices of any minimum monophonic graphoidal cover of \( G \), we have \( t_m \geq n + 1 \). Then by Theorem 2.5, \( \eta_m(G) \geq n + 1 \). Hence \( \eta_m(G) = n + 1 \).

Case (iv): \( k \geq 2 \) and there exists two non-adjacent vertices of degree greater than \( 2 \) on \( C \).

Let \( u, v \) be vertices of degree greater than \( 2 \) on \( C \) such that all vertices in a \((u-v)\)-section of \( C \) other than \( u, v \) have degree \( 2 \). Let \( P \) denote this \((u-v)\)-section and let \( G'' \) be the subgraph obtained by deleting all the internal vertices of \( P \). Clearly \( G'' \) is a tree with \( n \) pendant vertices and hence by Corollary 2.6, \( \eta_m(G'') = n - 1 \). If \( \psi''_m \) is a minimum monophonic graphoidal cover of \( G'' \), then \( \psi''_m \cup \{P\} \) is a monophonic graphoidal cover of \( G \) and hence \( \eta_m(G) \leq n \). Also, since \( G \) has \( n \) pendant vertices, \( t_m \geq n \) so that \( \eta_m(G) = n \).

Case (v): \( k \geq 3 \) and all the vertices of \( C \) are of degree greater than \( 2 \).

Let \( H = G - \{v_1v_2, v_2v_3\} \). Let \( H' \) and \( H'' \) be the components of \( H \) with \( H' \) contain the vertices \( v_1, v_3 \) and \( H'' \) contains the vertex \( v_2 \). Let \( r \) be the number of pendant vertices in \( H' \) and let \( s \) be the number of pendant vertices in \( H'' \). Since any pendant vertex of \( H' \) or \( H'' \) is a pendent vertex of \( G \), we have \( n = r + s \). Let \( G' = H' \) and \( G'' = H'' \cup \{v_1v_2, v_2v_3\} \). Then \( G' \) contains \( r \) pendant vertices and \( G'' \) contains \( s + 2 \) pendant vertices. Clearly \( G' \) and \( G'' \) are trees and hence by Corollary 2.6, \( \eta_m(G') = r - 1 \) and \( \eta_m(G'') = s + 1 \). Let \( \psi'_m \) be a minimum monophonic graphoidal cover of \( G' \) and let \( \psi''_m \) be a minimum monophonic graphoidal cover of \( G'' \). Then \( \psi'_m \cup \psi''_m \) is a monophonic graphoidal cover of \( G \) and hence \( \eta_m(G) \leq r - 1 + s + 1 = n \). Also, since \( G \) has \( n \) pendant vertices, \( t_m \geq n \) so that \( \eta_m(G) = n \).

We have seen that if \( G \) is a connected graph of order \( p \geq 3 \), then \( q - p \leq \eta_m(G) \leq q \). Also we have \( \eta_m(G) = q - p \) if and only if \( G \) has no simplicial vertices and there exists a monophonic graphoidal cover \( \psi_m \) such that every
vertex of $G$ is an internal vertex of a unique monophonic path in $\psi$ and $\eta_m(G) = q$ if and only if $G$ is complete. Also, it is proved that $\eta_m(G) = q - 1$ if and only if $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. In the following theorem, we give an improved bounds for the monophonic graphoidal covering number of a graph in terms of its size and monophonic diameter.

**Theorem 2.14.** For any connected graph $G$ of order $p \geq 2$, 
$$\left\lceil \frac{q}{d_m} \right\rceil \leq \eta_m(G) \leq q - d_m + 1,$$ 
where $d_m$ is the monophonic diameter of $G$.

**Proof.** Let $\psi_m$ be a minimum monophonic graphoidal cover of $G$. Since every edge of $G$ is in exactly one monophonic path in $\psi_m$, we have $q = \sum_{P \in \psi_m} |E(P)|$. Since $|E(P)| \leq d_m$ for each $P$ in $\psi_m$, we have $q \leq \eta_m(G).d_m$. Hence $\eta_m(G) \geq \left\lceil \frac{q}{d_m} \right\rceil$. Let $Q$ be a monophonic diametral path of $G$. It is clear that $\{(E(G) - E(Q)) \cup Q\}$ is a monophonic graphoidal cover of $G$. Hence $\eta_m(G) \leq |E(G) - E(Q)| + 1 = q - d_m + 1$. 

Now we give a realization result for the monophonic graphoidal covering number with some suitable conditions.

**Theorem 2.15.** For any positive integer $n$ with $q - p + 2 \leq n \leq q - p + 1$, there exists a tree $T$ such that the monophonic graphoidal covering number is $n$.

**Proof.** Let $P : v_1, v_2, v_3, ..., v_{q-n+2}$ be a path of order $q - n + 2$. Let $T$ be a tree obtained from $P$ by adding $n - 1$ new vertices $u_1, u_2, ..., u_{n-1}$ and joining each vertex $u_i (1 \leq i \leq n - 1)$ to the vertex $v_{q-n+1}$. The tree $T$ is given in Figure 2.3 and it has $n + 1$ pendant vertices. Then by Corollary 2.6, $\eta_m(T) = n$. 

![Figure 2.3: T](image)

**Remark 2.16.** In a tree $T$, $q = p - 1$ and so $q - p, q - p + 1$ are non-positive numbers. Hence there does not exist a tree $T$ whose monophonic graphoidal covering number is either $q - p$ or $q - p + 1$. Also, by Theorem 2.9, $\eta_m(G) = q$ if and only if $G$ is complete. Thus there does not exist a tree with the monophonic graphoidal covering number is $q - p$ or $q - p + 1$, or $q - 1$.
Problem 2.17. For any positive integer $n$ with $q - p \leq n \leq q$, does there exist a connected graph $G$ such that $G$ is not a tree and the monophonic graphoidal covering number is $n$?

References


